Fair Go Penny

Teacher Notes & Answers

7 8 9 **10** 11 12

TI-Nspire Investigation Student 40 min

Teacher Notes: In this activity students explore what appears to be a straight forward game of 'heads or tails' that turns out to be very different! The activity highlights the importance of thinking deeply about probability and understanding statistics. Students are introduced to the 'normal distribution' through the collection of sample means including an option to consider Type I and Type II errors. With more and more statistical evidence being portrayed in the media, it is important that students understand the information and are able to critically evaluate such evidence.

The wagers offered on the game are designed to increase interest and include things like 'expected return'; an important component of any gambling related scenario.

Investigation

Alex challenges his friend Gustav to a simple two player game that involves tossing a coin. Player A nominates a head / tail sequence, consisting of three outcomes. Player B can then nominate their head / tail sequence consisting of three consecutive outcomes. The coin is tossed repeatedly until one of the nominated outcomes is achieved.

Gustav doesn't seem interested in playing, he claims: "It's a silly game… if I choose Head, Tail, Head (HTH), it has a 1/8 chance of coming up, it doesn't matter what sequence you choose, it too will have a 1/8 chance of coming up." Alex is keen to play the game so he offers a wager as an incentive: "What if we play for money, best of 10 games, $2.00 per win" says Alex. Gustave agrees to play so the two boys make their selections:

Gustav chooses the sequence: Head, Tail, Head. (HTH)

Alex then chooses the sequence: Head, Head, Tail. (TTH)

Question: 1.

If the boys play 10 games who do you think will end up with the most money?

Answer: Answers will vary. The question asks students to think, it is not expected that students will already know the answer. If students base their logic on Gustav's comments, they will indeed get the answer wrong. Furthermore, students that do not understand event independence will most likely draw invalid conclusions. This question is worth discussing as a class and recording student misconceptions.

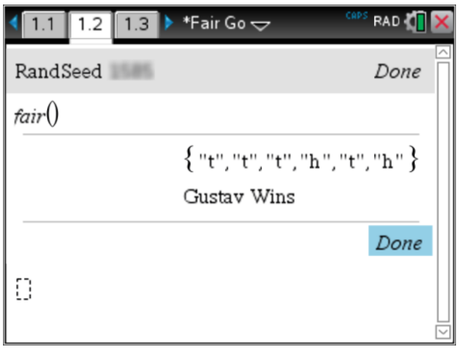
The game can be simulated on the calculator.

Instructions:

Open the TI-nspire file: "Fair Go" and Navigate to page 1.2.
A **single** game can be played. The results will be displayed as follows:

Event – List of coin toss results until a win was obtained.

Outcome – Name of winner



Author: P. Fox

Before commencing **any** probability simulation, it is important to 'seed' the random variable generator. The random seed command can be obtained from the calculator application:

> **Probability** > **Random** > **Seed**

Enter a unique four digit number such as the last four digits of your home or mobile number to ensure your results are also unique. To play a single game, press the [**VAR**] key, select the game and press [**ENTER**].

Question: 2.

Play the game 10 times and record the results in a table, A when Alex wins and G for Gustav. Alex's Balance is based on receiving $2.00 from Gustav if Alex wins or a $2.00 payout each time Gustav wins. Determine which player (if any) is the winner and comment on the results.

Answer: Answers will vary, but if students record their random variable seed, the results may be reproduced. In this game the results are in Alex's favour, so most students will likely see Alex with the majority of wins. The results shown below were generated for a random seed of 1234.

The sample results show that Alex won 70% of the games. If the game is fair there is a 17.2% chance that Alex will win at least 7 games.

Teacher Notes: It is worthwhile sharing results amongst the class. Consider first asking students to vote on whether or not they think the game is fair after this sample of 10 games. Some students may already be convinced with this small sample that the game is unfair, others may be unchanged even if Alex wins 8 games.

After collecting student responses about the fairness of the game, collect results from the class with regards to how many students had Alex winning the majority* of the time. This question is slightly different as it removes the 'by how much'. Remember at this stage we are trying to collect evidence about whether or not the game is 'fair'.
*Majority – A good question to ask students here is what to do about the 50:50 case?

Game	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Winner	A	A	A	G	G	A	A	A	G	A
Alex's Balance	$2	$4	$6	$4	$2	$4	$6	$8	$6	$8

Playing Detective

When the boys played, Gustav only won 3 games and subsequently claimed: "This game is not fair!" Playing 10 games does not constitute significant evidence to support Gustav's claim, his experience may be attributed to *chance*. Mathematicians make claims based on clearly defined boundaries and a level of certainty rather than reactionary emotions. One way to increase the level of confidence to accept or refute Gustav's claim is to conduct more trials (games).

Gustav decides to hire you as his 'maths-detective'. Your task is to determine whether Alex's game is fair. If the game is fair, the long term expected payout for both players is approximately $0.00, that is to say, neither player should *expect* to walk away with more money than their opponent.

As the mathematics detective, you decide the easiest way to investigate the fairness of the game is to simply play more games. In the real world, data collection generally costs money. The more data you collect the more it is likely to cost. Keep this in mind for the next part of the investigation. When prompted to simulate 'lots of games', consider what is reasonable enough to provide the evidence rather than simulating millions of games, which by the way will take the calculator a VERY long time!

Navigate to page 2.1 and select the simulation: fairmore() from the **[VAR]** menu.

This simulation requires you to select the number of games to be played, individual game results will not be displayed, rather the total number of games won by each player and Alex's balance based on the $2.00 payout each time Gustav wins and a $2.00 collection when Alex wins.

Question: 3.

Run the fairmore() simulation and select an appropriate number of games that you believe would need to be played in order to determine whether the current scenario is fair.

i) How many games did you choose to simulate, justify your selection.

Answer: Answers will vary, but should typically lie between 100 and 1000. As referenced in the text, data collection costs money, so 1,000,000 games is not a reasonable selection.

ii) What was the outcome? Do you believe the game to be fair under its current payout system?

Answer: Example: 100 game simulation: Alex won 63 and Gustav 37. Some students will be convinced by this result, others will accept that this could still have occurred by chance, which indeed it could. In another 100 game simulation, Alex wins 57 and Gustav wins 43, definitely not sufficient to 'prove' that the game is unfair.
Example: 1000 game simulation: Alex wins 587 and Gustav 413.

Teacher Notes: What is interesting in these larger simulations (100+ games), it is not always 'clear' that the game favours Alex, however surveying the class will reveal that Alex wins overall in the majority of cases. This is the 'introduction' to the concept of a sampling distribution.
To further this concept, students may be provided with a 'sticky dot' to place on the board and show on a number line the quantity of wins by Alex in a 100 game simulation. Whilst a class set of sticky dots will not produce a normal distribution, it reinforces the concept of a sampling distribution.
Even more data can be collected if using TI-Navigator™. Students can simulate several 100 game events and record the results. In a class of 25 students, with each student simulating 4+ games, 100 data points can be collected via a quick-poll and shared amongst the class in a matter of minutes.

Question: 4.

Suppose 50 games represent one sample. If you repeated these samples over and over again. What would the distribution of the collected sample means look like? Explain.

Answer: Answers will vary, students should be encouraged to draw a graph. The purpose of the question is to see if students intuitively have an understanding that the results will be distributed around a central mean and that the distribution gets smaller and smaller as you move further from the mean. This understanding should be included in student explanations.

Author: P.Fox

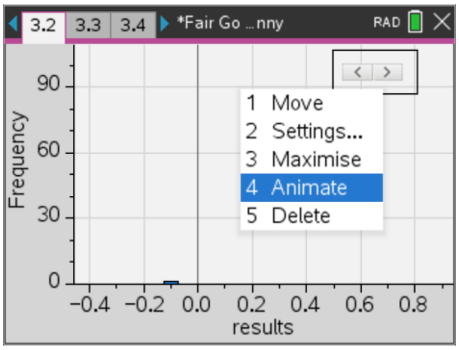
Navigate to page 3.1 and read the instructions. After reading the instructions, navigate to page 3.2.

Click on the slider a couple of times. Each time the slider is clicked another 50 (default value of SS) games are played and the average payout recorded.

Select the slider and press: ctrl + menu

From the contextual menu, select '**Animate**'.

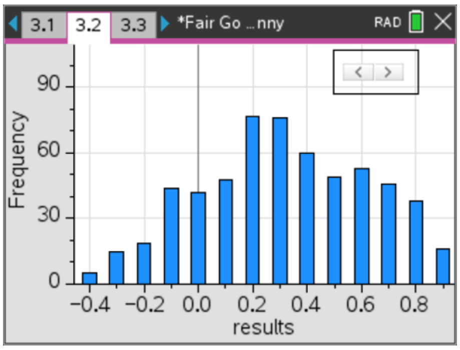
When one of the columns reaches approximately 50 repeat the above process and select **Stop** animation.

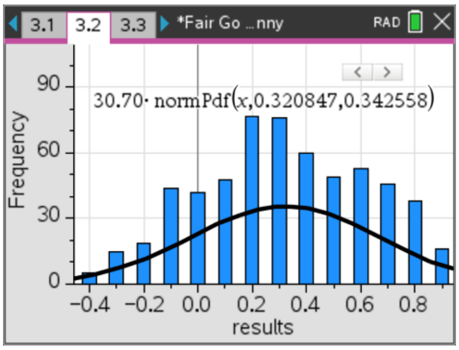


Question: 5.

How does the distribution of sample means (expected values) compare with your prediction in Question 4?

Answer: Answers will vary. Whilst the distribution generated will not be perfectly Normal, it will most likely resemble more frequent results in the centre and a reducing frequency as you move further from the mean.





Teacher Notes: It is possible to use the analyse menu to plot a normal distribution curve over the top of the data. The graph above shows the average 'expected return' per game for Alex is around 0.32. This means that from the 50 games in each sample, Alex should expect to walk away approximately $15.00 richer!
In the sample data (shown above) we can see that Gustav wins (overall) will only occur approximately 15% of the time, the game seems to be quite biased!

Question: 6.

As Gustav's detective, it's now time to provide him with your report. Write a brief report for Gustav identifying whether the game is fair.

Answer: Answers will vary. Students must refer to the data in their conclusion! Students may also explore some theoretical evidence to justify why the game is not fair. A relatively short tree diagram soon reveals some very damaging evidence. It is not expected that students will identify the theoretical probabilities, just that the game is not fair!

Extension: Checking for Errors

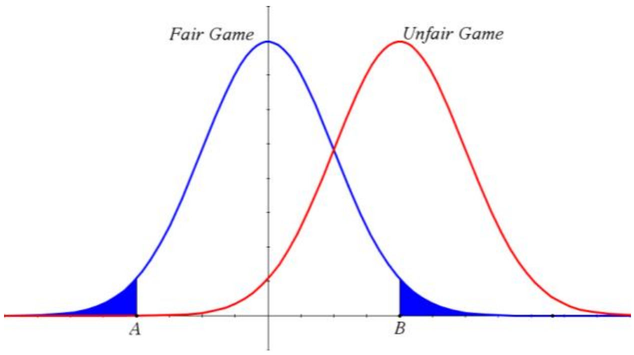
In a statistical context an error is different than a mistake! In statistics there are two basic types of errors:

Type I Reject the Null Hypothesis even though it is true.

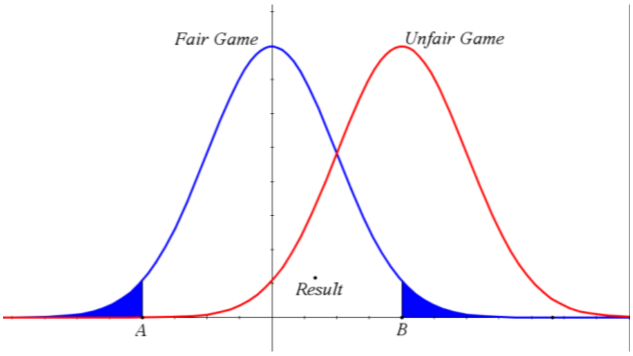
Type II Accept the Null Hypothesis even though it is not true.

As Gustav's detective we might start with the hypothesis that the game is fair, we would call this the Null Hypothesis. Gustav jumped to the conclusion that the game was unfair based on his experience, his data. Gustav rejects that the game is fair, he rejects the Null Hypothesis. If the game is fair then we could describe this as a Type I error. On the other hand, if our data did not provide sufficient evidence of any bias in the game, we would conclude the game is fair, but maybe the game is not actually fair. The following information helps explain how these errors occur.

The two graphs show the distribution of wins for a fair game (blue graph) and unfair game (red graph). The shaded regions under the blue graph (A & B) represent unlikely results, such as Alex wins 9 games out of 10 (B) or Gustav wins 9 games out of 10 (A). We know that these events can happen by chance in a fair game. What if we don't know if the game really is fair?



The second pair of graphs includes a 'result'. This may correspond to Alex winning 6 games from 10 in a fair game. On the other hand, it could represent 6 wins from 10 in an unfair game where Alex is expected to win 8 out of 10 games!



If we don't know if the game is fair, the result shown seems 'reasonable' enough to belong to the fair game. Indeed, it appears more likely to have come from the fair game than the unfair game as it is closer to the expected result (middle).

If it turns out the game is unfair, we will have experienced a Type II error. We accepted the game as fair even though it was not. Now imagine that the 'result' was in the blue shaded area (B). This is the time when we get suspicious and may accept that the result is more likely to have come from an unfair game. It is however possible that we are mistaken and jumping to conclusions. If we conclude the game is unfair, based on our result, but it did actually occur by chance, this would be a Type I error.

Question: 7.

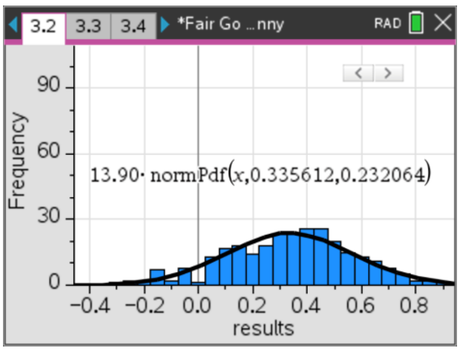
Generating larger samples makes the graphs shown above 'narrower'. Explain how this might help reduce the chances of jumping to the wrong conclusion (Type I or Type II error).

Answer: Students should include a diagram to help with their explanation. Reducing the overlap between the two graphs reduces the likelihood of a Type I or Type II error.

Question: 8.

Clear the data from spreadsheet on page 3.4. Change the sample size (SS) value on page 3.1 to 100 and run the simulations again. Explain how changing the sample size (number of games in each sample) will help determine whether the game is fair or not.

Answer: When the sample size is changed from 50 to 100 the 'deviation' away from the mean changes. We see less data falling into the region that we might consider 'fair' (belonging to the fair graph). A sample set of data is shown opposite. The mean and standard deviation of this graph provide numerical evidence, the data shows visual evidence that with the larger samples we see much less games falling into the category of what we might accept as fair.



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Teacher Notes: The title of this activity hides the real name. A quick Google search for Penney Ante.

The team at Numberphile have produced a wonderful video on this problem:

https://www.youtube.com/watch?v=Sa9jLWKrX0c

If students search for “Penny’s Problem” on something like YouTube to find an explanation, they are more likely to find videos from the Big Bang theory!

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