NUMB3RS Activity: To C or Not to C
Episode: "The Art of Reckoning"

Topic: Game theory – the Prisoner's Dilemma **Grade Level:** 9 - 12
Objective: To expose students to the iterative reasoning process of game theory.
Time: 15 - 20 minutes

Introduction

In "The Art of Reckoning," a convicted killer offers to give the FBI a detailed confession if they will let him visit his daughter. Don wants to pry information from him without giving too much in return. Charlie suggests that they use a game theory strategy called "tit-for-tat," from the game known as *Prisoner's Dilemma*. *Prisoner's Dilemma* gets its name from the hypothetical case where two people are arrested for a crime. They are interrogated separately and have no contact with each other. Each is told that he faces a sentence of y years in prison. However, if he gives up information on the other prisoner for a larger crime, his sentence will be reduced to x years and the other prisoner will serve z years (where $x < y < z$). But if each prisoner gives the other up, *both* will serve the z-year sentence. In this activity, students will test different strategies for the game *Prisoner's Dilemma*.

Discuss with Students

Call the players A and B. If they "cooperate" with each other and do not give up the other prisoner, they will *both* serve y years. If they both "defect" and give up information on each other, they will *both* serve z years. If A defects and B cooperates, then A will get off with only x years while B will have to serve z years, or vice versa.

It is in each individual's best interest to defect, given that the other person cooperates. Both players benefit by cooperating with each other. However, the worst case is if they both defect. Since they have no contact with each other, this is the dilemma each player faces.

In an actual case, each prisoner would get only one chance. But in theory, game theorists have made it into a game where the possible outcomes have point values. Play continues for a number of rounds, and the player with more total points wins. There are many strategies for playing, but nobody has ever proven that one strategy is the best. Charlie recommends one simple strategy, "tit-for-tat," that has been very successful in tournaments against far more sophisticated methods.

In this activity, students first play against one another using any strategy of their own design, and then compare the strategies at the end. Next, one plays using the "tit-for-tat" strategy. If one player knows when the last round is coming, that person has an advantage and could change strategies. Therefore, the game is generally played with a random number of moves. This number is not revealed to the players. For this activity, the teacher should stop each game so that students cannot anticipate the end.

Student Page Answers:

1. *Answers will vary. For example, one player might defect every time, hoping to maximize points but change when the player realizes that it does not work.* **2.** *Answers will vary. For example, it may be better to take any punishment rather than risk the death penalty—death could be thought of as having a value as bad as $-\infty$.* **3.** *and* **4.** *Answers will vary. In general, "tit-for-tat" will probably do better. It is also possible that one or both players came up with "tit-for-tat" in their first game.* **5.** *The game will be a tie, but one that maximizes the "common good."* **6.** *Player A wins if the game goes an even number of turns longer; otherwise it is a tie.* **7.** *Both players start with "tit-for-tat." Then, one player switches to play just C. After playing C for several turns, switch to D and use it for the remainder of the game. This will ensure a (small) win.*

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Name: ________________________________ Date: _______________

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In "The Art of Reckoning," a convicted killer offers to give the FBI a detailed confession if they will let him visit his daughter. Don wants to pry information from him without giving too much in return. Charlie suggests that they use a game theory strategy "tit-for-tat," from the game known as *Prisoner's Dilemma.*

Prisoner's Dilemma gets its name from the hypothetical case of two criminals who are arrested and held separately from each other. The police have enough information to indict them on a smaller crime, but suspect they are behind a larger crime. Each is told that if one testifies against the other, he or she will get a sentence less than that for the smaller crime, while the other will get a longer sentence for the larger crime. In *Prisoner's Dilemma*, "cooperate" means that the prisoner *does not* give up the other prisoner; "defect" means that the prisoner *does* give up the other prisoner. These terms indicate that the prisoners cooperate with each other or defect from one another, not that they cooperate with the investigators.

In an actual case, each prisoner would get only one chance. But in theory, game theorists have made it into a game where the possible outcomes have point values. Play continues for a number of rounds, and the player with more total points wins. There are many strategies for playing, but nobody has ever proven that one strategy is the best. This activity provides the opportunity to experiment with a strategy.

In this activity, the payoff is given as points (not years in prison), so the object is to get as many points as possible. This table shows the point values:

		Player B	
		Cooperates	**Defects**
Player A	**Cooperates**	A gets 3 B gets 3	A gets 1 B gets 5
	Defects	A gets 5 B gets 1	A gets 1 B gets 1

Each player needs a copy of this chart:

TURN	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D
B	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D	C D

Each player should secretly decide on a strategy. For each turn, select either C (cooperate) or D (defect) by circling the appropriate letter so that the opponent cannot see the chart, and there is no clue from hearing how the letter is written. Reveal the choices simultaneously and decide the points from the payoff table. You should also record your other opponent's choice in order to identify his/her strategy.

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If a player knows when the game will end, there is an advantage because his/her strategy could be changed for the last turn. It is important that the game is ended by an outsider. Therefore, your teacher or a classmate will tell you when the game ends.

1. Play the above game with a partner before reading on. The teacher will announce when to end the game (the table allows for 20 turns, but the number can be adapted as needed). Discuss the strategies used.

2. How could the strategy be changed for a different set of point values? For example, what value could be given to the cooperating player if when the other player defects, the cooperating player has all points taken away? Why?

Charlie points out that in competition, a rather simple strategy has been used to defeat some of the most complicated strategies. This simple strategy is called “tit-for-tat.” To use this strategy, start with C, and then for each subsequent turn, copy the opponent’s choice from the *previous* move. The player “rewards” the opponent who cooperates by cooperating on the next move and “punishes” the opponent who defects by defecting on the next move. Nobody has ever proved that this is the best strategy, but statistically, it has dominated competitions.

3. Play another round with Player A using the same strategy as the first time, but this time Player B should use “tit-for-tat.” Describe the results of this game.

4. Play a third round with A and B switching roles from Question 3. Who wins this time?

5. What is the result if both players use the “tit-for-tat” strategy?

6. What is the result if both players use “tit-for-tat,” but on one move somewhere in the game, player A defects when the strategy calls for cooperation?

7. If both players begin with “tit-for-tat,” what strategy should a player change to in order to guarantee a victory?

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

Game theory is a branch of mathematics that has many more important and far-reaching applications than the name may imply, such as in warfare and business. Mathematician John von Neumann (who was recently honored on a U.S. postage stamp) is credited as one of the creators of this branch of mathematics in the first half of the 20th century. As Charlie indicates, research and applications are very much alive in today's world.

For the Student

In the movie "A Beautiful Mind," Princeton math professor John Nash is shown using game theory to maximize the potential of getting dates for his group of students. He then immediately realizes that this has greater potential and rushes home to write up his ideas. These later become the basis of his work that eventually won him the Nobel Prize in economics. Find examples of the influence that game theory has had in many fields. Applications of game theory can be surprising.

Find an explanation of *Nash equilibrium*. Imagine different situations in which this concept applies.

Do research in the use of game theory in other fields (labor negotiations in economics, the President vs. the Congress in government, colonies of animals where individuals sacrifice their own self-interest for the common good in biology, etc.) Write a report showing the connection to mathematics.

In "The Art of Reckoning," Charlie also discusses two other strategies. The first is called the "grim trigger" strategy. A player cooperates every time until the first time that the opponent defects. From that point on, the player *never* cooperates again. The second strategy is a variation of the one used in this activity, called "tit-for-two-tats," in which the player copies what the opponent did two moves ago. Try the game with these strategies.

Additional Resources

There are competitions in *Prisoner's Dilemma* in which participants program computers with a strategy and then are pitted against each other. As Charlie noted, the simple "tit-for-tat" has consistently outperformed these more sophisticated approaches.

Prisoner's Dilemma can be played online in many places. For example, try playing at the following site and see if you can determine the strategy the computer is using.
http://pespmc1.vub.ac.be/PRISDIL.html

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