Vertex and Factored Forms of Quadratic Functions

Math Objectives

- Students will be able to identify and justify the effect of changing the parameters of a quadratic function in vertex and factored form.
- Students will be able to identify and justify the relationship between the factored form and *x*-intercepts of a quadratic function.
- Students will be able to identify and justify which form is more appropriate to use when solving a given problem.
- Students will make sense of problems and persevere in solving them. (CCSS Mathematical Practice)
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)

Vocabulary

- factored form of a quadratic function
- vertex form of a quadratic function
- vertex
- *x*-intercepts (zeroes, roots)

About the Lesson

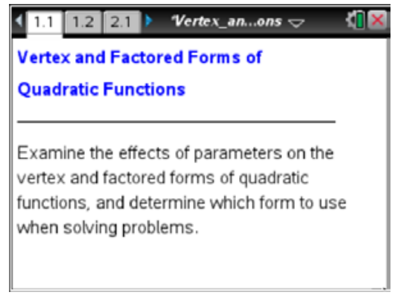
- This lesson involves utilizing sliders on a quadratic function expressed in vertex and factored forms to determine the effect the parameters have on the graph of the function. Students will:
  - Manipulate sliders and make and validate conjectures about the relationships between the values of a, h, and k in the function $f(x) = a(x - h)^2 + k$ and the location of the vertex.
  - Manipulate sliders and make and validate conjectures about the relationships between the values of a, r, and s in the function $f(x) = a(x - r)(x - s)$ and the location of the zeroes.
  - Determine which form of a quadratic function would be most appropriate to solve specific classes of problems.

TI-Nspire™ Navigator™

- Use Live Presenter to demonstrate how to utilize sliders.
- Use Class Capture to monitor students' progress.
- Use Quick Poll to assess students' understanding.

Activity Materials

Compatible TI Technologies: + TI-Nspire™ CX Handhelds, + TI-Nspire™ Apps for iPad®, + TI-Nspire™ Software



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Click on sliders to change the parameters of a function

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.

Lesson Files:

Student Activity
Vertex_and_Factored_Forms_of_Quadratic_Functions_Student.pdf
Vertex_and_Factored_Forms_of_Quadratic_Functions_Student.doc
TI-Nspire document
Vertex_and_Factored_Forms_of_Quadratic_Functions.tns

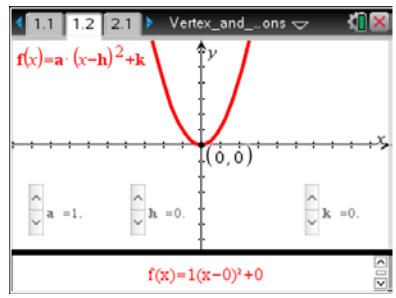
Visit www.mathnspired.com for lesson updates.

Discussion Points and Possible Answers

Teacher Tip: A TI-Nspire document for students' use is included with this activity. However, if you or your students wish to create your own TI-Nspire document(s) to learn how to incorporate sliders in your lesson, see the accompanying Create document for this activity.

TI-Nspire Navigator Opportunity: *Class Capture/Live Presenter*
See Note 1 at the end of this lesson.

Move to page 1.2.



1. Given the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, Sam said that a change in the value of k results in a change in the y-coordinate of each point on the graph. Do you agree or disagree with Sam? Use the sliders to investigate. Explain your reasoning.

   <u>Answer:</u> Sam's reasoning is correct. Changes in the value of k move the current graph into a resulting graph that is k units higher or lower depending on the value of k.

2. Sal observed that when $f(x) = 1(x - 3)^2$, all of the x-coordinates are 3 less than they were when $f(x) = 1x^2$. Do you agree or disagree with Sal? Use the sliders to investigate. Explain your reasoning.

   <u>Answer:</u> Disagree. The x-values for the changed function have not changed, but the changed function has caused the graph to be horizontally translated to the right 3 units.

   Teacher Tip: You may want to help students see this by first graphing $f(x) = 1x^2$, which is the same thing as $f(x) = 1(x - 0)^2 + 0$.

3. Change slider a and describe its effect on the parabola. Discuss the effect of the sign of a (whether it is positive or negative), its magnitude (how big or small it is), and anything else that seems important.

   <u>Answer:</u> If a is positive, the parabola opens upward; if a is negative, the parabola opens downward. The larger the absolute value of a, the narrower the parabola. The closer a is to zero, the wider the parabola. All points on the parabola are changed except for the vertex.

Teacher Tip: You may want to have some discussion as to what happens if $a = 0$.

4. Given the function $f(x) = a(x - h)^2 + k$, describe in general what effect changing h will have on the graph of the parabola. What does it have to do with the vertex? Use the sliders to investigate if necessary. Explain your answer.

   Answer: The parabola translates right and left as h changes (right as h gets bigger, left as it gets smaller). The value of h is the x-coordinate of the vertex.

5. Given the function $f(x) = a(x - h)^2 + k$, describe in general what effect changing k will have on the graph of the parabola. What does it have to do with the vertex? Use the sliders to investigate if necessary. Explain your answer.

   Answer: The parabola translates up and down as k changes (up as k gets bigger, down as it gets smaller). The y-coordinate of the vertex is the k value.

Teacher Tip: You may want to help students see that the vertex is located at the point (h, k), which is why this form of the function is called vertex form.

6. Using the form $f(x) = a(x - h)^2 + k$, describe the graph and the function that has a vertex of $(-2, -5)$. Is there more than one answer?

   Answer: From the origin, the parabola will be translated left 2 units and down 5 units. There is more than one answer. Since the a value can be positive or negative, the parabola could open upward ($a > 0$) or downward ($a < 0$) and could also be vertically stretched or compressed. The function is $f(x) = a(x + 2)^2 - 5$.

Teacher Tip: When working with the function $f(x) = a(x - h)^2 + k$, students generally have little difficulty seeing that the value of k shifts the parabola up k units when $k > 0$ and down k units when $k < 0$.

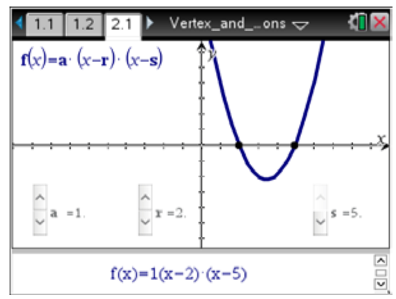
However, students often have a very difficult time with the effect of h. You may want to spend some time on this concept as students work through this question.

Perhaps point out that the vertex of the parent function, $f(x) = x^2$, has a vertex at the point (0, 0). Ask students to determine the vertex of the translated function $f(x) = (x – h)^2$. At the end of your discussion, students should be able to look at a function such as $f(x) = (x – 5)^2 – 7$ and realize that the vertex of the parabola is (5, –7).

TI-Nspire Navigator Opportunity: *Quick Poll (Equations)*
See Note 2 at the end of this lesson.

Move to page 2.1.

On this page, there is another form of the quadratic, the factored form: $f(x) = a(x – r)(x – s)$.



TI-Nspire Navigator Opportunity: *Class Capture/Live Presenter*
See Note 3 at the end of this lesson.

7. Change slider *a* to change the value of the variable. Suzy thinks that as the *a*-value gets larger, the parabola will be stretched away from the *x*-axis, and as the *a*-value gets smaller, it will be compressed toward the *x*-axis. Is her thinking accurate? Explain. Does a change in the value of *a* have the same effect as it did in the vertex form?

   Answer: Yes, a change in the value of *a* has the same effect. If $a > 0$, the parabola opens up, and if $a < 0$, the parabola opens down. The value of *a* also determines whether the graph has been vertically stretched or compressed from the parent function: $f(x) = x^2$. If $|a| > 1$, the function is stretched away from the *x*-axis. If $0 < |a| < 1$, the function is compressed toward the *x*-axis.

8. Changes in the value of *a* seem to result in changes in all the points on the graph except for two: the *x*-intercepts of the parabola (the roots or zeroes). Adjust all the sliders and observe the effect that each has on the *x*-intercepts. How are the locations of the *x*-intercepts related to the values of the sliders?

   Answer: When *a* is changed, the two *x*-intercepts remain fixed. When the values of *r* and *s* are changed, the location of the x-intercepts change. One *x*-intercept matches the value of *r* while the other matches the value of *s*.

9. Jason said that changing the value of *r* moves the parabola horizontally. Jeremy said that changing the value of *s* also moves the parabola horizontally. Who is correct? Why? What other information do the *r* and *s* values provide?

Sample answers: Because *r* and *s* are being subtracted from the *x*-value, they move the parabola horizontally. However, their role is much more important than merely shifting the parabola horizontally. The values of *r* and *s* determine the location of the zeroes of the function. Given the function $f(x) = a(x – r)(x – s)$, the zeroes are located at $x = r$ and $x = s$.

> **Teacher Tip:** When working with the function $f(x) = a(x – r)(x – s)$, students generally have little difficulty seeing that when *r* and *s* are both positive, the zeroes of the function are located at $x = r$ and $x = s$, both positive numbers. When one or both parameters are negative, students sometimes start to get confused. Be sure that they see functions such as $f(x) = (x – 3)(x + 4)$ to remind them about the double negative (x – -4) and that the zeroes are located at $x = 3$ and $x = –4$.

> **Teacher Tip:** You may want to help students see that this form of the quadratic function is called factored form or intercept form.

10. In factored form, what seems to be the relationship between the vertex and the *x*-intercepts? Write an expression for the *x*-coordinate of the vertex in terms of *r* and *s*.

Answer: The vertex of the parabola lies midway between the two zeroes of the quadratic function. The *x*-coordinate of the vertex is the average of the values of *r* and *s*, i.e., $x = \frac{r+s}{2}$. The *y*-coordinate of the vertex can be obtained by substituting the *x*-value into the given quadratic function.

11. Change the sliders so that $r = s$. Describe the resulting parabola.

Answer: The resulting parabola has two equal roots and has its vertex on the *x*-axis. The *x*-value of the vertex is the same as the value of the two equal roots.

12. Write a quadratic function with zeroes at $x = –2$ and $x = 3$. Use the form $f(x) = a(x – r)(x – s)$ and change the sliders to check your function.

Answer: When you substitute $r = –2$ and $s = 3$, you get the function $f(x) = a(x + 2)(x – 3)$. Using $a = 1$, students may give the function $f(x) = (x + 2)(x – 3)$. However, you may receive functions with other values of *a*. If so, you may want to take this opportunity to discuss the effect of changing the value of *a* and the fact that there are an infinite number of possible functions.

Teacher Tip: This is a good opportunity for you to probe students about why an infinite number of functions can be created. The *x*-coordinate of the vertex is determined, but not the *y*-coordinate. This will help students see that two points do not determine a unique parabola.

TI-Nspire Navigator Opportunity: *Quick Poll (Equations)*
See Note 4 at the end of this lesson.

Teacher Tip: For additional discussion on the effect of changing the value of *a*, refer to the activity *Standard Form of Quadratic Functions.*

13. Three different forms for a quadratic function are:

Standard form: $f(x) = 3x^2 + 6x - 24$
Vertex form: $f(x) = 3(x + 1)^2 - 27$
Factored form: $f(x) = 3(x + 4)(x - 2)$

a. Show that the three forms are equivalent.

Answer: To show that the three forms are equivalent, transform any of the three forms into any of the other forms. However, you will expand the vertex and factored forms to standard form.

Vertex form:
$$\begin{aligned} f(x) &= 3(x + 1)^2 - 27 \\ &= 3(x^2 + 2x + 1) - 27 \\ &= 3x^2 + 6x + 3 - 27 \\ &= 3x^2 + 6x - 24 \end{aligned}$$

Factored form:
$$\begin{aligned} f(x) &= 3(x + 4)(x - 2) \\ &= 3(x^2 + 2x - 8) \\ &= 3x^2 + 6x - 24 \end{aligned}$$

b. Determine each of the following and explain how to choose the best form of the quadratic function for obtaining your answer.

- the smallest value(s) of the function

  Answer: To find the smallest value of the function, you must find the *y*-coordinate of the vertex. Since the parabola opens up ($a = 3$), you know the vertex is the minimum point on the graph. By looking at the vertex form of the function, you see that the smallest value of the function is $y = -27$.

- the *x*-value(s) or the zero(s) of the function

  Answer: To find the zeroes of the function, you can look at the factored form. Set $f(x) = 0$ and obtain $0 = 3(x + 4)(x - 2)$. Thus, $x = -4$ and $x = 2$ are the zeroes of the function.

- the value(s) of the function when $x = 0$

Answer: To find the values of the function when $x = 0$, use standard form and replace all of the x-values with $x = 0$. Thus, $f(0) = –24$. Note that, although a quadratic function can have two x-intercepts, it can only have one y-intercept. (Otherwise, it would not be a function.)

TI-Nspire Navigator Opportunity: *Quick Poll (Multiple Choice)*
See Note 5 at the end of this lesson.

14. A ball is thrown up in the air. Three different forms for the height of the ball, in feet, as a function of time, x, in seconds, are:

Standard form: $f(x) = -16x^2 + 32x + 48$

Vertex form: $f(x) = -16(x – 1)^2 + 64$

Factored form: $f(x) = -16(x – 3)(x + 1)$

a. Show that the three forms are equivalent.

Answer: To show that the three forms are equivalent, you will again expand the vertex and factored forms.

Vertex form:
$$\begin{aligned} f(x) &= -16(x-1)^2 + 64 \\ &= -16(x^2 - 2x + 1) + 64 \\ &= -16x^2 + 32x - 16 + 64 \\ &= -16x^2 + 32x + 48 \end{aligned}$$

Factored form:
$$\begin{aligned} f(x) &= -16(x-3)(x+1) \\ &= -16(x^2 - 2x - 3) \\ &= -16x^2 + 32x + 48 \end{aligned}$$

b. Determine each of the following and explain how to choose the best form of the quadratic function for obtaining your answer.

- the time for the ball to hit the ground

Answer: To find the time for the ball to hit the ground, you have to find when the height is zero. This will occur at the zeroes of the function. You will use the factored form and set $f(x) = 0$. Thus, $0 = -16(x-3)(x+1)$. Solve to get $x = 3$ and $x = –1$. Since time cannot be negative, you know that the ball hits the ground three seconds after it is thrown.

- the time for the ball to reach its maximum height

Answer: To find the time for the ball to reach its maximum height, you must find the x-coordinate of the vertex. You will thus use the vertex form. Since the parabola opens down ($a = –16$), you know the vertex is the maximum point on the graph. Looking at the function $f(x) = -16(x – 1)^2 + 64$, you see that the x-coordinate of the vertex is $x = 1$. Thus, the ball reaches its maximum height in one second.

Alternatively, you can take the function $f(x) = -16(x - 1)^2 + 64$ and compare it to the vertex form of $f(x) = a(x - h)^2 + k$ to see that $h = 1$ and $k = 64$. You can then see that the vertex is located at the point (1, 64). Thus, $x = 1$, and you know that the ball reaches its maximum height in one second.

Also, you can take the function in factored form and see that the zeroes are at –1 and 3. The vertex lies midway between its two zeroes. The x-coordinate is $\frac{-1+3}{2} = 1$. Thus, $x = 1$, and you know that the ball reaches its maximum height in one second.

- the initial height from which the ball was thrown

Answer: To find the initial height from which the ball was thrown, you must find the height at time $x = 0$. This is the y-intercept. By looking at the standard form $f(x) = -16x^2 + 32x + 48$ and setting $x = 0$, you see that $f(0) = 48$. Thus, the ball is thrown from an initial height of 48 feet. (Apparently, the person throwing the ball was either on the fourth floor of a building or standing on a platform.)

- the maximum height of the ball

Answer: To find the maximum height of the ball, you look again at the vertex form of the function. This time, you must look at the y-coordinate of the vertex, since the height of the ball is represented by the y-value. You see that $y = 64$. Thus, one second after it was thrown, the ball reaches its maximum height of 64 feet.

Alternatively, you can take the function $f(x) = -16(x - 1)^2 + 64$ and compare it to the vertex form of $f(x) = a(x - h)^2 + k$ to see that $h = 1$ and $k = 64$. You can then see that the vertex is located at the point (1, 64). Thus, $y = 64$, and you know that, in one second, the ball reaches its maximum height of 64 feet.
Or, you can take the function in factored form, find the x-value of the vertex, substitute this value in the function, and find the y-value of the vertex.

TI-Nspire Navigator Opportunity: *Quick Poll (Multiple Choice)*
See Note 6 at the end of this lesson.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- The effect of changing the parameters of a quadratic function in vertex form.
- The effect of changing the parameters of a quadratic function in factored form.
- The appropriate form to use when solving problems.

Assessment

Throughout the lesson, check that students understand the relationship between the form of the function and the graph of the function. Questions 6 and 12 ask students to write a function to produce the given result. Check that students obtain the correct function using Quick Poll as described below. In Quick Poll, students could be asked to show their work to answer questions 13a and 14a. The last two questions of the activity give students the opportunity to see when and how to utilize the various forms of quadratic functions. It is important to ensure that students understand the purpose of each of the three forms of the function.

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Note 1

Question 1, *Class Capture/Live Presenter:* You may want to use *Live Presenter* to have a student demonstrate the procedure for using sliders to change the parameters in the function. Use *Class Capture* to monitor students' progress throughout the activity.

Note 2

Question 6, *Quick Poll (Equations):* Before moving on to factored form, you may want to assess students' understanding of vertex form by sending them an *Equations Quick Poll*. Have them type in a few functions in vertex form using a variety of vertices. If students are having difficulty writing the functions, you may want to review the material covered in the preceding questions.

Note 3

Question 7, *Class Capture/Live Presenter:* You may want to use *Live Presenter* to have a student demonstrate the procedure for using sliders to change the parameters in the function. Continue to use *Class Capture* to monitor students' progress throughout the rest of the activity.

Note 4

Question 12, *Quick Poll (Equations):* You may want to take this opportunity to assess students' understanding of factored form by sending them an *Equations Quick Poll*. Have them type in a few functions in factored form using a variety of zeroes. Take some time to assure that students have mastered the concepts.

Note 5

Question 13b, *Quick Poll (Multiple Choice):* Tell students that you are going to send a *Multiple Choice Quick Poll*. Ask them to choose which of the three forms of the quadratic function they feel would be best to use for each of the given scenarios. Assign the letters A through C to the functions and ask students to choose their answer. When you receive the results, ask students to share their reasons for making their choices.

Note 6

Question 14b, *Quick Poll (Multiple Choice):* Send another *Multiple Choice Quick Poll*. Assign the letters A through C to the functions and ask students which of the three forms of the quadratic function they feel would be best to use for each of the given scenarios. When you receive the results, ask students to share their reasons for making their choices.