VECTOR AND PARAMETRIC MODELS OF PROJECTILE MOTION

by C. C. Edwards
Coastal Carolina University, Conway, SC
Edwards@coastal.edu
(Please feel free to email me questions and /or comments.)

Key Topic: Parametric, Vector and Polar Functions

Abstract:

In this activity we derive the vector model of projective motion, and then we derive the parametric model from the vector model. An example is then supplied which shows the student how to solve a typical projectile motion problem using the TI-89.

This activity is appropriate for students who have a good understanding of parametric, but only a very basic knowledge of vectors. If students have had a mild exposure to vectors prior to taking calculus can investigate this activity after they have learned the fundamentals of parametric equations.

Prerequisite Skills:

- Ability to integrate
- Basic understanding of parametric equations
- Very basic knowledge of vectors: length, horizontal and vertical components

Degree of Difficulty: Easy to moderate

Needed Materials: TI-89

NCTM Principles and Standards:

- Content Standards – Algebra
    - Represent and analyze mathematical situations and structures using algebraic symbols
    - Use mathematical models to represent and understand quantitative relationships
    - Draw a reasonable conclusion about situation being modeled

- Process Standards
    - Representation
    - Connections
    - Problem Solving

VECTOR AND PARAMETRIC MODELS OF PROJECTILE MOTION

On page 378 of the ***TI-89 Guidebook*** there is a very interesting application which investigates the flight of a baseball which is hit at an angle of 32 degrees with an initial velocity of 95 feet per second. After telling you how to set the **MODE** in the TI-89, it then tells you to enter the parametric equations $xt1(t) = 95t\cos(32^\circ)$ and $yt1(t) = -16t^2 + 95t\sin(32^\circ)$. Then it shows you how to investigate the position of the ball by using trace and looking at a table of values. But it never told you how it got these parametric equations!

But that's OK. The ***Guidebook*** is designed to tell you how to use your calculator. It's not a textbook which teaches you mathematics. It assumes that you know where these equations come from. This activity will not only show you where these equations came from, but it will also show you that the ***Guidebook*** is assuming that the ball is being hit at ground level!

Let's look at the general situation:

> **An object is launched at a distance d from the ground. It is launched at an angle θ with an initial speed s.**

Our goal is to model this situation. We will first do this using vectors.

It is an established physical observation that, for objects near the surface of the earth, the force exerted on this object by gravity is:

$$\mathbf{F} = -mg\mathbf{j}$$

where m is the mass of the object and $g = 32\ \text{ft/s}^2 = 9.8\ \text{m/s}^2$ is the gravity constant. And by Newton's Second Law of Motion (ignoring air resistance)

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{a}$ is the acceleration vector of the object. This gives us:

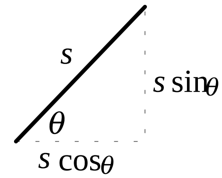
$$\mathbf{a} = -g\mathbf{j}\,.$$

By integrating, we find that the velocity vector is

$$\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$$

where $\mathbf{v}_0$ is the initial velocity vector.

Since the object was launched at an angle θ with an initial speed s, simple trigonometry tells us that



$$\mathbf{v}_0 = (s\cos\theta)\mathbf{i} + (s\sin\theta)\mathbf{j}\,.$$

So $$\mathbf{v}(t) = (s\cos\theta)\mathbf{i} + (-gt + s\sin\theta)\mathbf{j}\,.$$

And again by integrating, we find that the position vector is

$$\mathbf{r}(t) = (s\cos\theta)t\mathbf{i} + \left[-\tfrac{1}{2}gt^2 + (s\sin\theta)t\right]\mathbf{j} + \mathbf{r}_0\,.$$

But the object was launched at a distance d from the ground. So $\mathbf{r}_0 = d\mathbf{j}$.

Hence $$\mathbf{r}(t) = (s\cos\theta)t\mathbf{i} + \left[-\tfrac{1}{2}gt^2 + (s\sin\theta)t + d\right]\mathbf{j}\,.$$

This gives us the vector model for projectile motion. And from this we can easily see that the parametric model is

$$x(t) = st\cos\theta$$
$$y(t) = -\tfrac{1}{2}gt^2 + st\sin\theta + d$$

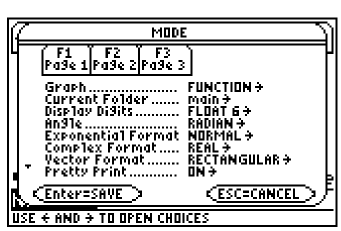
where $s =$ the speed at which the object was launched,
$\theta =$ the angle at which the object was launches, and
$d =$ the distance from the ground at which the object was launched.

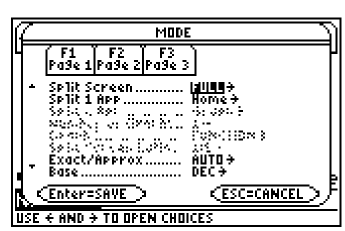
EXAMPLE. A baseball is hit 3.5 feet from the ground at an angle of 32 degrees and an initial velocity of 95 feet per second.

a. How high does the object go?
b. How far does the object travel horizontally?
c. What is the speed of the object when it hits the ground?

We will now show you how to solve this problem using a TI-89. ***Whenever you start a new problem, clear memory by pressing*** [2nd][F6]**2.**

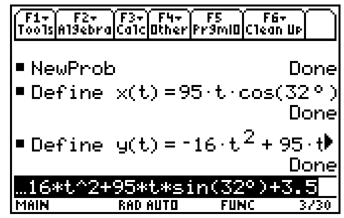
Set the first two pages of the **MODE** as shown at the right.





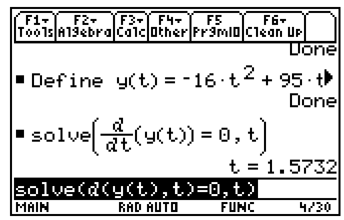
Since distance is measured in feet, we will use $g = 32$ ft/s^2. And from the statement of the problem, $d = 3.5$, $s = 95$, and $\theta = 32^\circ$. So our parametric equations are

$$x(t) = 95t\cos(32^\circ)$$
$$y(t) = -16t^2 + 95t\sin(32^\circ) + 3.5$$

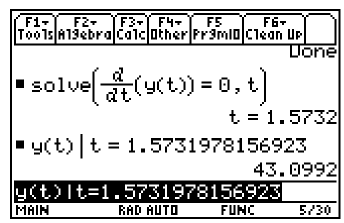


Enter the equation for $x(t)$ by pressing
[F4][ENTER][X][(][T][)][=][9][5][×][T][×]
[2nd][COS][3][2][2nd][°][)][ENTER].
In a similar fashion, define the equation for $y(t)$.

Since $y(t)$ denoted the distance of the ball from the ground, the ball reaches its maximum height at the time when $y'(t) = 0$.



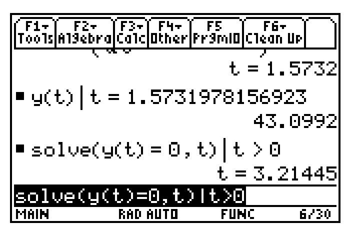
Solve the equation $y'(t) = 0$ by pressing
[F2][ENTER][F3][ENTER][Y][(][T][)][,][T][)][=][0][,][T][)][ENTER].



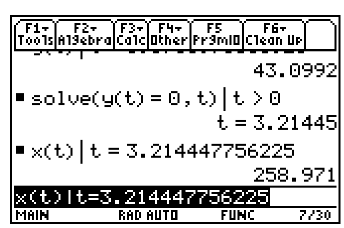
Find the maximum height by plugging this solution into the height function.
[Y][(][T][)][|][▲][ENTER][ENTER].

We see that the maximum height is approximately 43 feet.

The maximum horizontal distance is attained when the ball hits the ground, i.e., when $y(t) = 0$. But $y(t)$ is a quadratic equation, so we will get two solutions for t.



Solve $y(t) = 0$ for the positive solution of t by pressing
[F2][ENTER][Y][(][T][)][=][0][,][T][)]
[|][T][2nd][MATH][8][1][0][ENTER]

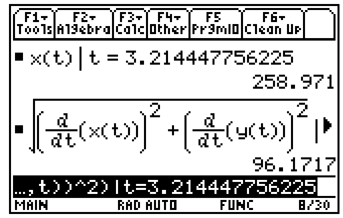


Then find this horizontal distance by plugging this value for t into the equation for $x(t)$.

We see that the maximum horizontal distance is approximately 259 feet.

In the vector model, the position vector was found to be $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. So the velocity vector is $\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$. Hence the speed at time t is

$$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$



Find the speed when the ball at the time the ball hits the ground by pressing [2nd][√][(][F3][ENTER][X][(][T][T][,][T][)][)][^][2][+][(][F3][ENTER][Y][(][T][T][,][T][)][)][^][2][)][|][▲][▲][▲][ENTER][ENTER].

From this we see that the ball hits the ground at a speed of approximately 96 feet per second.