EXPLORATIONS

Activity 11

Get Your Numbers in Shape

Objectives

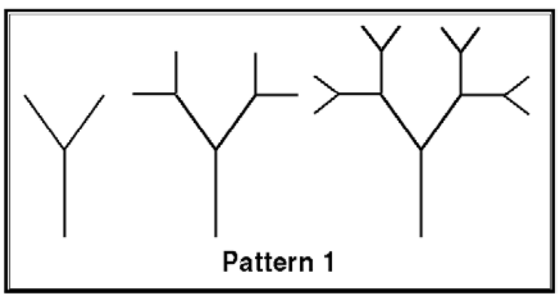
- To use technology to explore patterns
- To use inductive reasoning to make conjectures about patterns
- To use technology to produce a sequence
- To find the *x* value of a function, given the *y* value
- To find the *y* value of a function, given the *x* value
- To find a linear or a quadratic equation for a given pattern

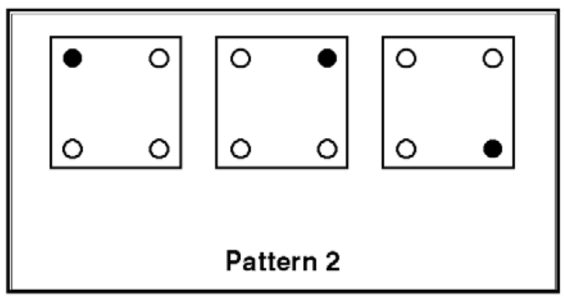
Materials

- TI-73 graphing device
- Small cubes or candy

Introduction

Study the pictures below and find the next shape in each of the patterns shown.





The process that you have used to find the next shape in each pattern is called *inductive reasoning*. You have used this type of reasoning since you were a baby. You learned how to eat using the proper utensils and how to turn on the television by observing others and drawing conclusions. After several trials, you perfected your skills. Inductive reasoning allows you to make generalizations based on a pattern of specific examples or past events. These generalizations are called *conjectures* or *hypotheses*. Mathematicians and scientists use inductive reasoning to make discoveries and develop formulas based on their discoveries.

Mathematicians have used number patterns to describe a variety of phenomena in nature. For example, a famous pattern that is found throughout nature is the *Fibonacci sequence* (see Activities 4 and 8.) In this activity, you will investigate a variety of sequences.

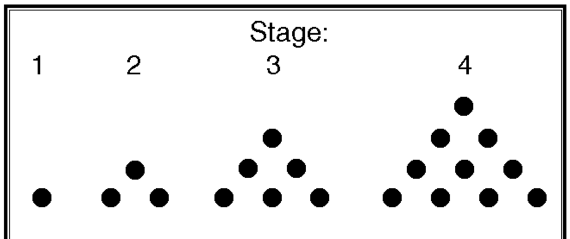
Problem

How do you use inductive reasoning to find a formula for the patterns that describe an array of objects or a sequence of numbers?

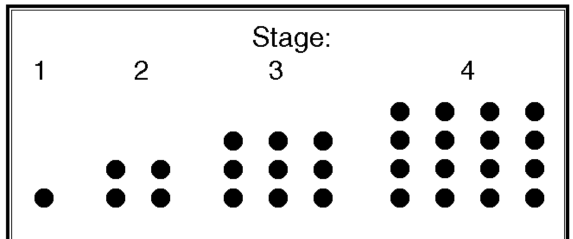
Collecting the data

A *triangular number* is a number that can be represented by a triangular arrangement of objects.

A *square number* is a number that can be represented by a square arrangement of objects. Numbers that correspond to geometric figures are called *figurate numbers*. The diagrams below illustrate triangular numbers and square numbers.



Triangular Numbers



Square Numbers

Use the objects that your teacher will give you to find the next three stages for the triangular numbers. Complete the table on the **Data Collection and Analysis** page.

Do you notice a pattern in the table? Suppose you needed to know the 100th triangular number. You could continue to form the pattern using your objects, however this could become tedious and time-consuming. If you knew a rule or *function* for calculating any term in this sequence, you could find the 100th term of the sequence with relative ease.

You will use the *finite differences* technique to find a function for this sequence. This method requires that you find the difference between successive values in the sequence. If the first differences are equal, then the function or rule that describes the sequence is *linear.* If the second differences are equal, then the function or rule that describes the sequence is *quadratic*. You can use the TI-73 to find the differences.

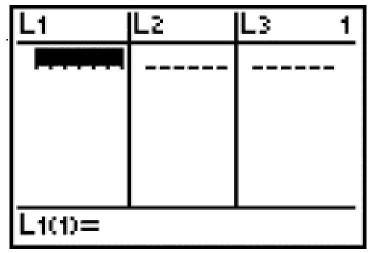
Setting up the TI-73

Before starting your data collection, make sure that the TI-73 has the STAT PLOTS turned OFF, Y= functions turned OFF or cleared, the MODE and FORMAT set to their defaults, and the lists cleared. See the Appendix for a detailed description of the general setup steps.

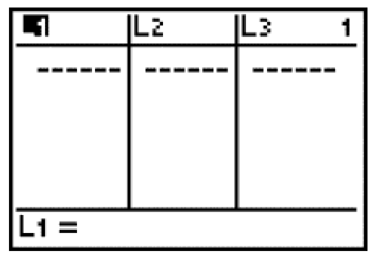
Entering the data in the TI-73

Enter the stage number data in **L1** and the value in **L2**. You will use the *sequence* command on the TI-73 to enter the data in **L1**.

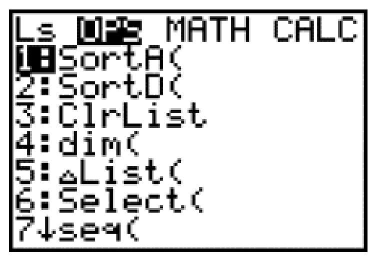
1. Press [LIST].



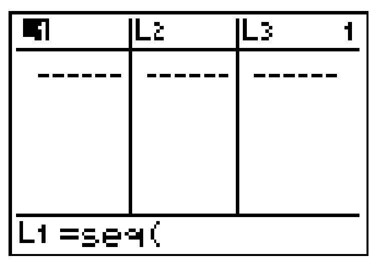
2. Press [▲] to highlight **L1**.



3. Press [2nd] [STAT] [▶] to move the cursor to the **OPS** menu.

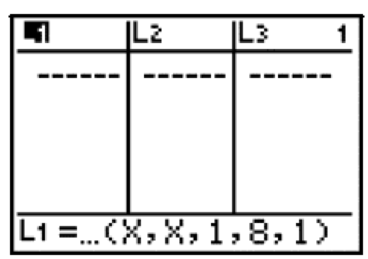


4. Select **7:seq(** by pressing **7**.

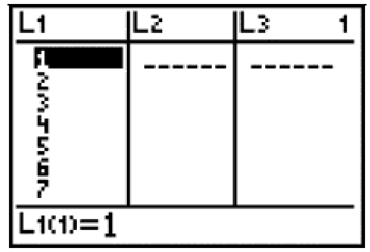


5. Press [x] [,] [x] [,] **1** [,] **8** [,] **1** [)] to complete the command.

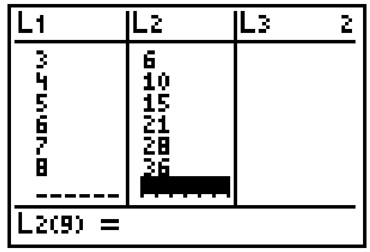
__Note__: The components of the sequence command are __seq(__ expression or formula, variable, beginning value, ending value, increment).



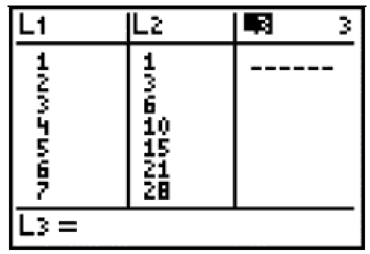
6. Press [ENTER]. The sequence of numbers from 1 to 8 will be placed in **L1**.



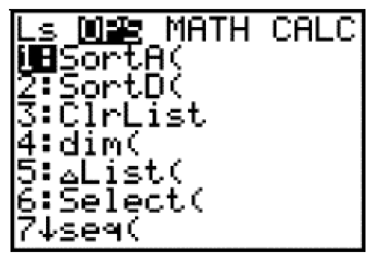
7. Enter the values for the first eight stages of the triangular numbers in **L2**.



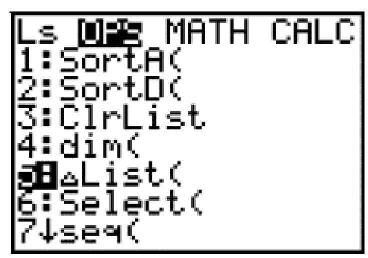
8. Find the first differences. Press [▶] [▲] to move the cursor to the top of **L3**.



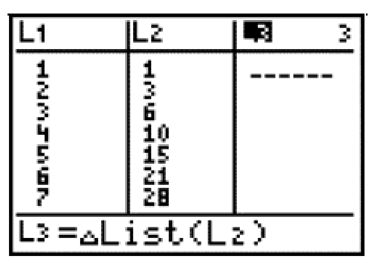
9. Press [2nd] [STAT] [▶] to move the cursor to the **OPS** menu.



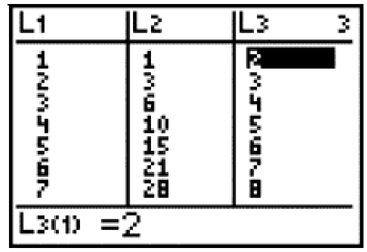
10. Select **5:ΔList(** by pressing **5**.



11. Press [2nd] [STAT] **2:L2** [)].

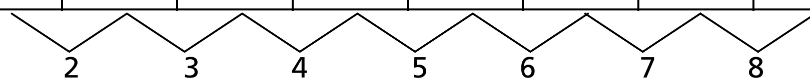


12. Press [ENTER] to see the difference between the successive terms in **L2**.

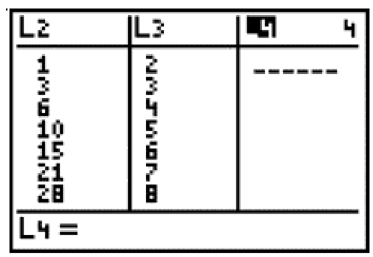


The numbers in **L3** represent the first difference of the sequence. Observe that the values are not equal. Therefore, the function that describes this sequence is not *linear*.

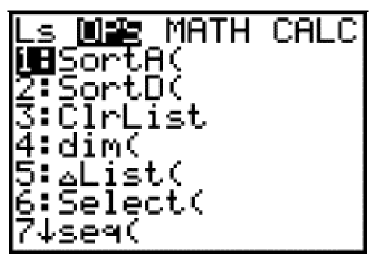
Term Number	1	2	3	4	5	6	7	8
Value	1	3	6	10	15	21	28	36



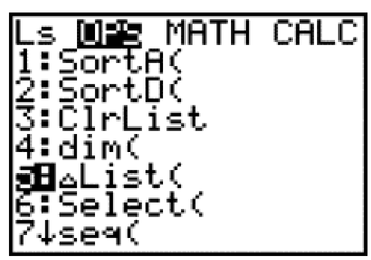
13. Find the second differences. Press [▶] [▲] to move the cursor to the top of **L4**.



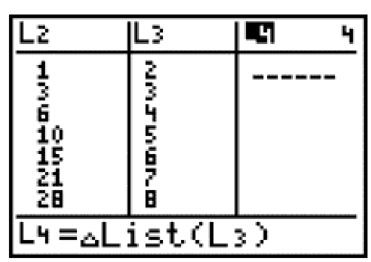
14. Press [2nd] [STAT] [▶] to move the cursor to the **OPS** menu.



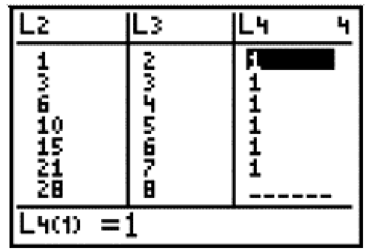
15. Select **5:ΔList(** by pressing **5**.



16. Press [2nd] [STAT] **3:L3** [)].



17. Press [ENTER] to see the difference between the successive terms in **L3**.

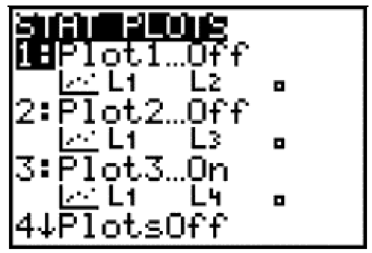


Since the second differences are equal, the function that describes this sequence is *quadratic.*

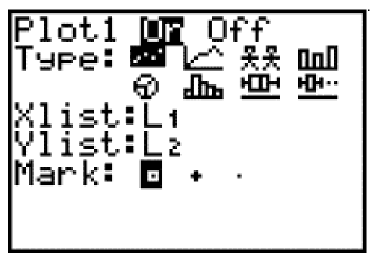
Graphing the data: Setting up a scatter plot

Plot the data using **L1** and **L2**.

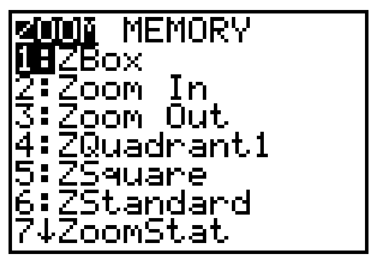
1. Press [2nd] [PLOT]. Select **1:Plot1** by pressing **1** or [ENTER].



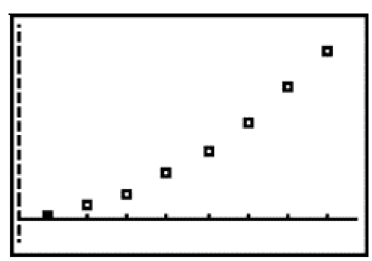
2. Set up the plot by pressing [ENTER] [▼] [ENTER] [▼] [2nd] [STAT] **1:L1** [▼] [2nd] [STAT] **2:L2** [▼] [ENTER].



3. Press [ZOOM].



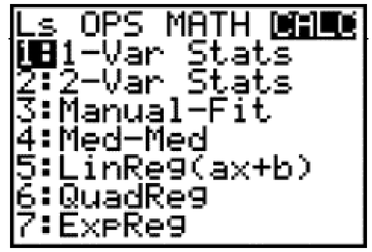
4. Select **7:ZoomStat** by pressing **7**.



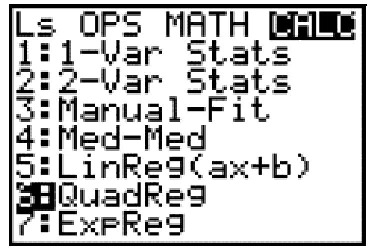
Analyzing the data: Finding a trend line

Use a quadratic regression to find the quadratic function that describes this sequence.

1. Press [2nd] [STAT] [◄] to move the cursor to the **CALC** menu.

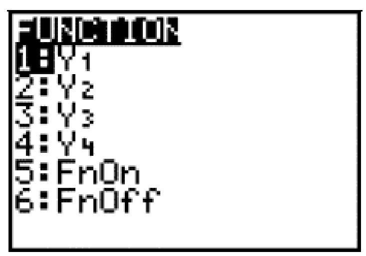


2. Select **6:QuadReg** by pressing **6**.



3. Press [2nd] [STAT] **1:L1** [,] [2nd] [STAT] **2:L2** [,].

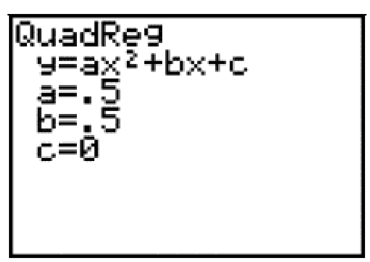
4. Press [2nd] [VARS]. Select **2:Y-VARS** by pressing **2**.



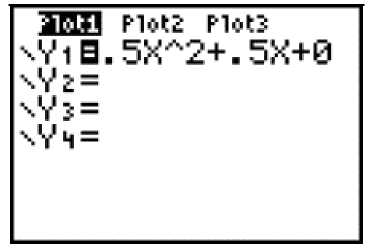
5. Select **1:Y1** by pressing **1** or [ENTER].

6. Press [ENTER] to see the function.

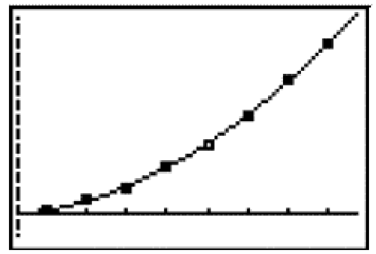
   The function that describes the sequence for triangular numbers has been pasted in **Y1**.



7. Press [Y=] to see the equation.



8. Press [GRAPH] to see the graph of the function.

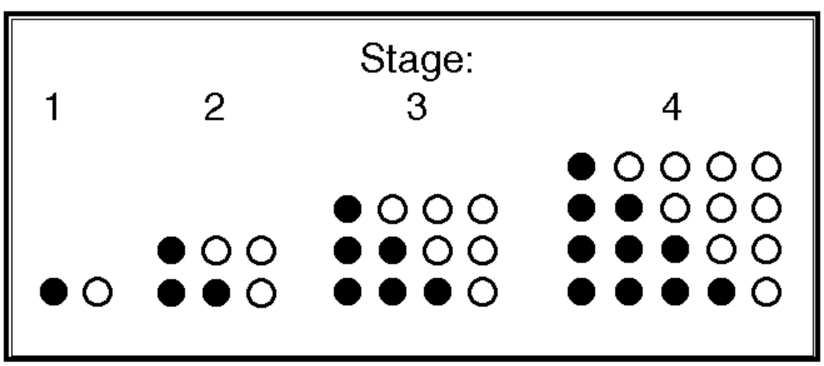


Use the function to answer Part I questions 1 through 4 on the **Data Collection and Analysis** page.

The formula for triangular numbers is a sequence. A *sequence* is an ordered list of numbers. You have graphed the formula in function mode. In function mode, values such as 1.5, 2.3, and 4.7 can be evaluated. For a sequence, however, the domain can only be integers. Therefore, the plot of the data is a better representation of the triangular numbers than is the graph of the function.

Finding the formula, a different approach

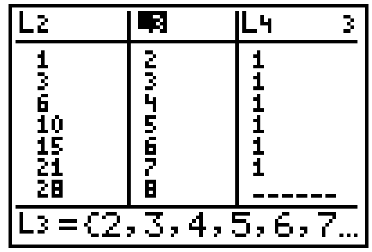
Another way to find a formula for the *triangular numbers* is illustrated below. Examine the diagram and table below for triangular numbers. The black dots represent a triangular number, and the white dots represent a second triangular number.



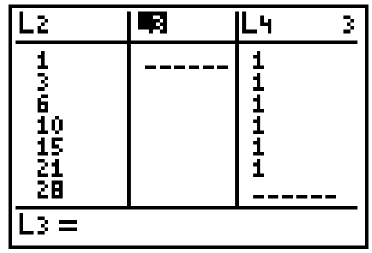
1. Term Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	***n***
2. Value	1	3	6	10	15	21	28	36	
3. Twice the Value	2	6	12	20	30	42	56	72	
4. Factors for Row 3	**1** • 2	**2** • 3	**3** • 4	**4** • 5	**5** • 6	**6** • 7	**7** • 8	**8** • 9	$\boldsymbol{n}(n+1)$

Since the formula represents twice the actual triangular numbers, you must divide by 2 to find the formula for the triangular numbers. The formula is: $y = \frac{1}{2} n(n + 1)$ or $y = .5\, n(n + 1)$. Compare this formula to the function in **Y1**.

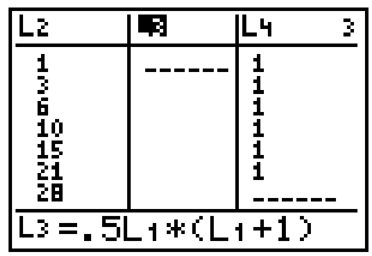
1. Press [LIST]. Move the cursor to the top of **L3**.



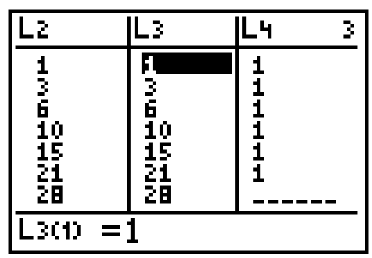
2. Press [CLEAR] [ENTER] to clear **L3**. Press [▲] to highlight **L3** again.



3. Enter the formula for the sequence, .5n(n+1), except substitute **L1** for *n*. Press [.] **5** [2nd] [STAT] **1:L1** [×] [(] [2nd] [STAT] **1:L1** [+] **1** [)].



4. Press [ENTER] to calculate the values. Compare the values found in **L2** and **L3**.



There are two types of formulas used to represent sequences. The type you have just found is an example of an explicit formula. (You will investigate the other type later in this activity.) An *explicit formula* is a formula that tells how to find the value of any term of a sequence without finding all the previous terms.

Complete the **Entering the data**, **Graphing the data**, and **Analyzing the data** sections using square numbers.

Answer questions 5 - 7 on the **Data Collection and Analysis** page.

Finding a different formula

As mentioned before, there are two different types of formulas used to represent sequences. The type you are going to investigate is called a recursive formula. A recursive formula is a formula that tells how to find a term using the previous term in the sequence. You will now investigate a recursive formula for triangular numbers.

Study the pattern listed below to see how the recursive formula is developed.

1. Term Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	***n***
2. Value	1	3	6	10	15	21	28	36	
3. Formula	**1**+ 0	**2** + 1	**3** + 3	**4** + 6	**5** + 10	**6** + 15	**7** + 21	**8** + 28	**?**

To develop the next term in the sequence, you must add the term number to the value of the previous term. Use **Next** to indicate the next value and **Previous** to indicate the previous value.

Next Value = Previous Value + Term Number

For example: Value of term 4 = Value of term 3 + Term Number

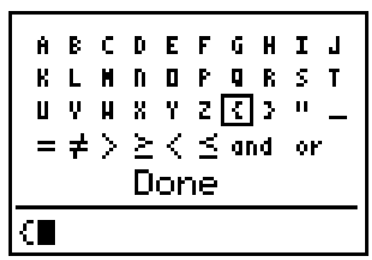
10 = 6 + 4

Note: 6 *is the value of the* 3rd *term and* 4 *is the term number.*

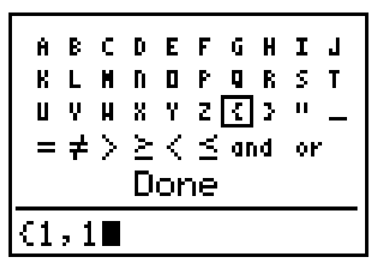
Using the TI-73 to test the formula

1. Generate the terms of the sequence. Press [2nd] [QUIT] to go to the Home screen. Press [CLEAR] to clear the Home screen.

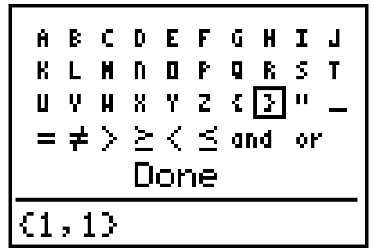
2. Press [2nd] [TEXT]. Press [▼] [▼] [◄] [◄] [◄] [◄] [◄] to highlight the opening brace, "{". Press [ENTER] to type "{".



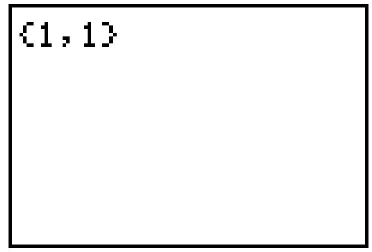
3. Press **1** [,] **1**. These are the term number and the value of the first term, respectively.



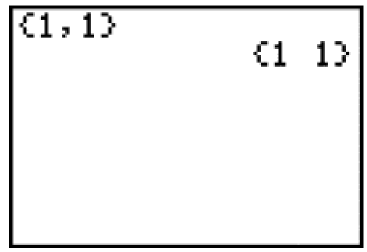
4. Close the expression by pressing ▸ ENTER to type the closing brace, "}".



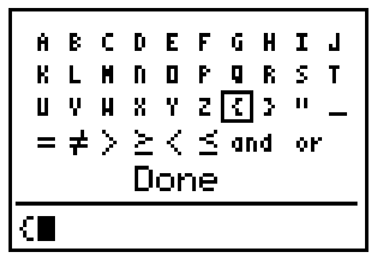
5. Press ▾ ▾ ENTER to exit the Text editor.



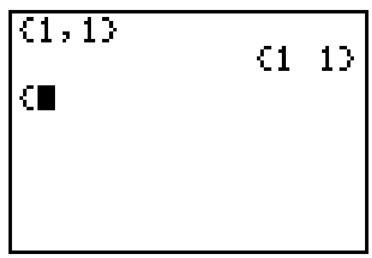
6. Press ENTER to store these values on the TI-73. We will refer to the term number as **Ans(1)**, and to the value of the term as **Ans(2)**.



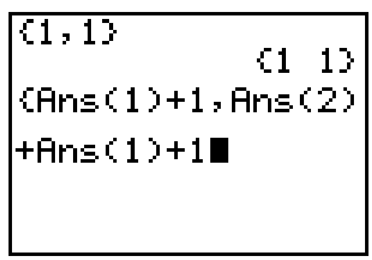
7. Press 2nd [TEXT]. Press ▾ ▾ ◂ ◂ ◂ ◂ ◂ to highlight the opening brace, "{". Press ENTER to type "{".



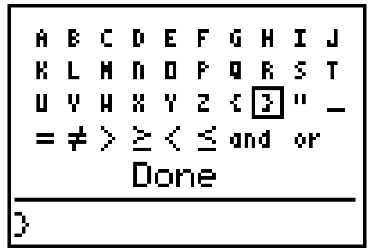
8. Press ▾ ▾ ENTER to exit the Text editor.



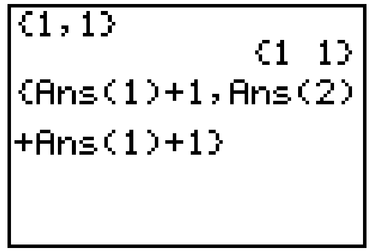
9. Press 2nd [ANS] (**1**) + **1** , 2nd [ANS] (**2**) + 2nd [ANS] (**1**) + **1.**



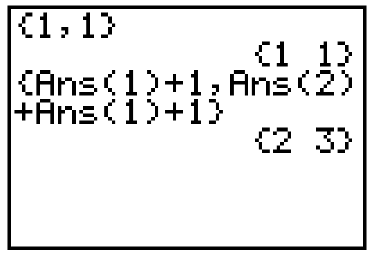
10. Press [2nd] [TEXT]. Press ▼ ▼ ◀ ◀ ◀ to highlight the closing brace, "}". Press [ENTER] to type "}".



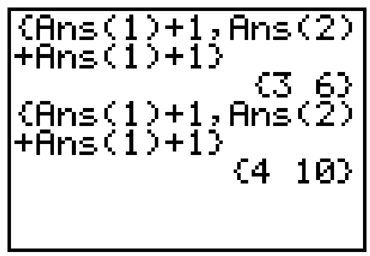
11. Press ▼ ▼ [ENTER] to exit the Text editor.



12. Press [ENTER] to generate the second term in the sequence.



13. Continue to press [ENTER] to generate successive terms in the sequence.



Answer questions 8 – 10 on the **Data Collection and Analysis** page.

Data Collection and Analysis

Name ____________________________

Date ____________________________

Activity 11: Get Your Numbers in Shape

Collecting the data

Record your data for triangular numbers in the table below.

Stage Number	1	2	3	4	5	6	7	8
Value	1	3	6	10				

Analyzing the data

Use your equation from number 8 in the **Analyzing the data: Finding a trend line** section to answer questions 1 through 4.

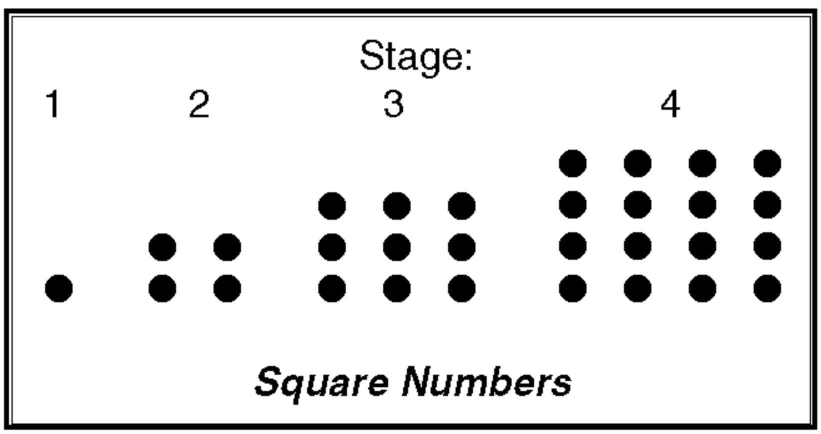
1. Find the 20th *triangular number*. ________________________________

2. Find the 100th *triangular number*. ______________________________

3. What is the stage for a *triangular number* that equals 120? ____________

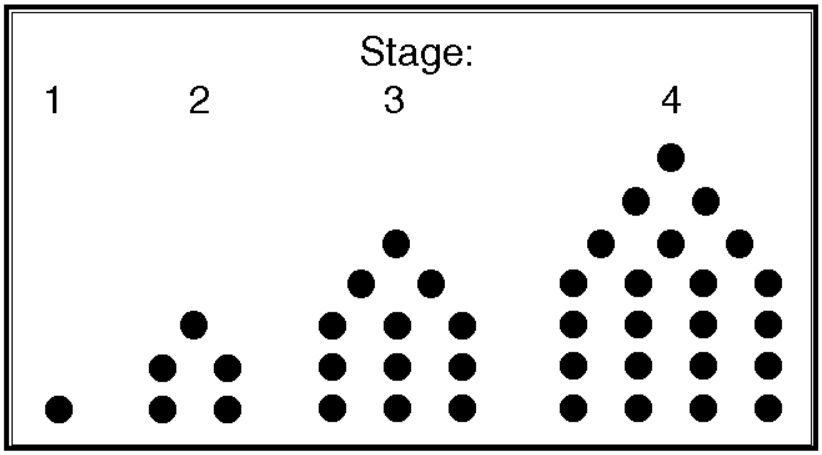
4. What is the stage for a *triangular number* that equals 496? ____________

5. Find an explicit formula for the *square numbers*. ____________________



6. Find an explicit formula for the pentagonal numbers shown below.

____



7. You go to a back-to-school party. Thirty of your friends are at the party. After the long summer break, you are happy to see all of your friends. You begin shaking hands with each person at the party. In fact, each person at the party shakes hands with everyone else at the party. How many handshakes are there altogether?

____

8. Did the values for the triangular numbers sequency match the values you found using the recursive formula? Explain.

____

____

____

9. Find a recursive formula for the square numbers.

____

10. Find a recursive formula for the pentagonal numbers.

____

Note**: For questions 9 and 10, follow the steps in the* ***Using the TI-73 to test the formula section *to check your work.*

Teacher Notes

Activity 11

Get Your Numbers in Shape

Objectives

- To use technology to explore patterns
- To use inductive reasoning to make conjectures about patterns
- To use technology to produce a sequence
- To find the *x* value of a function, given the *y* value
- To find the *y* value of a function, given the *x* value
- To find a linear or a quadratic equation for a given pattern

Materials

- TI-73 graphing device
- Small cubes or candy

Preparation

For questions 6 and 10, give the students the following hint.
Hint: Look at the dots. What two figurate patterns form the pentagonal pattern? Look at the formulas for these two patterns.

Management

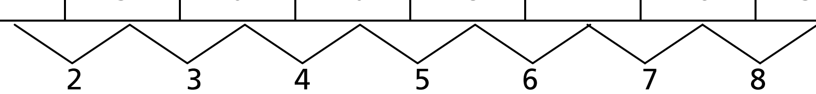
For question number 7, act out the situation starting with a smaller group of students. Develop a table to keep track of the results. Gradually increase the number of students until there are 20 students. To help the students visualize this, have students draw a polygon, where the number of vertices of the polygon represent the number of students.

Answers to Data Collection and Analysis

Collecting the data

Sample data:

Term Number	1	2	3	4	5	6	7	8
Value	1	3	6	10	15	21	28	36



Analyzing the data

Use your equation from number 8 in the **Analyzing the data: Finding a trend line** section to answer questions 1 through 4.

1. Find the 20th *triangular number*.

210

2. Find the 100th *triangular number*.

5050

3. What is the stage for a *triangular number* that equals 120?

15

4. What is the stage for a *triangular number* that equals 496?

31

5. Find an explicit formula for the square numbers.

$y = x^2$

6. Find an explicit formula for the pentagonal numbers.

$Y = 1.5x^2 - 0.5x$

7. You go to a back-to-school party. Thirty of your friends are at the party. After the long summer break, you are happy to see all of your friends. You begin shaking hands with each person at the party. In fact, each person at the party shakes hands with everyone else at the party. How many handshakes are there altogether? *formula:* $y = ½\, n(n - 1)$

For 31 people there are 465 handshakes.

8. Did the values for the triangular numbers sequence match the values you found using the recursive formula? Explain.

Yes. Both formulas give the same sequence of numbers.

9. Find a recursive formula for the *square numbers*.

For any term, n = previous term + (2n + 1)

10. Find a recursive formula for the pentagonal numbers shown below. $n(n + 1)$

For any term, n = previous term + (3n - 2)