

# First Order Linear Differential Equations

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Materials Needed: TI-Nspire CAS unit  
TI-Nspire Navigator  
FirstOrderLinearDE.tns file  
IntegratingFactorDerivation.tns file  
GeneralSolutionofFirstOrderLinearDE.tns file

Courses: AP Calculus BC  
Differential Equations

Objective: This activity will introduce students to first order linear differential equations.

Procedures: Send the FirstOrderLinear DE.tns file to the students.

The students should review pages 1.1 – 1.6 on their own. You should give students a few minutes to read these pages, and then discuss the information with them.

The image displays six screenshots of TI-Nspire Navigator pages, arranged in a 2x3 grid. Each screenshot shows a page from a document titled "First Order Linear Differential Equations".

- Page 1.1:** Shows the title "First Order Linear Differential Equations" and the author's name "Donald Worcester, Winter Park High School".
- Page 1.2:** Contains a "Definition:" section stating that a first order linear differential equation has the form  $\frac{dy}{dx} + P(x) \cdot y = Q(x)$ , where  $P(x)$  and  $Q(x)$  are continuous functions.
- Page 1.3:** Explains that the first order linear differential equation of the form  $\frac{dy}{dx} + P(x) \cdot y = Q(x)$  is said to be in standard form.
- Page 1.4:** States that it is time to derive the general solution to a first order linear differential equation and that an integrating factor  $u(x)$  is needed.
- Page 1.5:** Describes using the integrating factor to rewrite the left hand side of the equation into the derivative of the product of  $u(x) \cdot y$ .
- Page 1.6:** Shows the step one process of rewriting the left hand side as  $u(x) \cdot \frac{dy}{dx} + u(x) \cdot P(x) \cdot y$  and setting it equal to  $\frac{d}{dx}(u(x) \cdot y)$ .

Students should enter their solution for  $u(x)$  on page 1.7.

1.7 1.8 1.9 1.10 RAD AUTO REAL

After deriving the integrating factor, type your solution for  $u(x)$  below.

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Once all of the students have derived (or attempted to derive) a value of  $u(x)$ , you should send all students the IntegratingFactorDerviation.tns file and review the derivation with them.

1.5 1.6 1.7 1.8 RAD AUTO REAL

Before continuing with the activity, please wait for your instructor to review the derivation of the integrating factor.

You will want to use the correct integrating factor when completing the rest of this activity.

1.1 RAD AUTO REAL

$$u(x) \cdot \frac{dy}{dx} + u(x) \cdot P(x) \cdot y = \frac{d}{dx}(u(x) \cdot y)$$

$$\rightarrow u(x) \cdot y' + u(x) \cdot P(x) \cdot y = u(x) \cdot y' + y \cdot u'(x)$$

$$\rightarrow u(x) \cdot P(x) \cdot y = y \cdot u'(x)$$

$$\rightarrow u(x) \cdot P(x) = u'(x)$$

$$\rightarrow P(x) = \frac{u'(x)}{u(x)}$$

$$\rightarrow \int P(x) dx = \int \left( \frac{u'(x)}{u(x)} \right) dx$$

1.1 RAD AUTO REAL

$$\rightarrow u(x) \cdot P(x) = u'(x)$$

$$\rightarrow P(x) = \frac{u'(x)}{u(x)}$$

$$\rightarrow \int P(x) dx = \int \left( \frac{u'(x)}{u(x)} \right) dx$$

$$\rightarrow \int P(x) dx + C = \ln|u(x)|$$

$$\rightarrow e^{\int P(x) dx} + C = e^{\ln|u(x)|}$$

$$\rightarrow C \cdot e^{\int P(x) dx} = u(x)$$

Students will now find a general solution to a first order linear differential equation. Students should review pages 1.9 – 1.10, and then enter their general solution on page 1.11.

1.6 1.7 1.8 1.9 RAD AUTO REAL

Let  $c=1$  and find a general solution to the first order linear differential equation, by multiplying the general form by  $u(x)$ .

$$\left( \frac{dy}{dx} + P(x) \cdot y = Q(x) \right) \cdot u(x)$$

1.7 1.8 1.9 1.10 RAD AUTO REAL

At this time, derive the general solution to the first order linear differential equation.

1.9 1.10 1.11 1.12 RAD AUTO REAL

After deriving the general solution to the first order linear differential equation, type your solution below.

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Once all of the students have derived (or attempted to derive) a general solution to the first order linear differential equation, you should send all students the GeneralSolutionofFirstOrderLinearDE.tns file and review the derivation with them.

The first screenshot shows the differential equation  $y'e^{\int P(x)dx} + y \cdot P(x) \cdot e^{\int P(x)dx} = Q(x) \cdot e^{\int P(x)dx}$  and its simplified form  $\frac{d}{dx}(y \cdot e^{\int P(x)dx}) = Q(x) \cdot e^{\int P(x)dx}$ . A note states: "To get the results in the second line above, we needed to notice the important use of the product rule."

The second screenshot shows the integration step:  $\int \frac{d}{dx}(y \cdot e^{\int P(x)dx}) = \int (Q(x) \cdot e^{\int P(x)dx}) dx$ , resulting in  $y \cdot e^{\int P(x)dx} = \int (Q(x) \cdot e^{\int P(x)dx}) dx + C$ .

The third screenshot shows the final general solution:  $y = \frac{1}{e^{\int P(x)dx}} \left( \int (Q(x) \cdot e^{\int P(x)dx}) dx + C \right)$ . A note states: "Therefore, we can state that the general solution of a first order differential equation of the form  $y' + P(x) \cdot y = Q(x)$  is given by  $y \cdot e^{\int P(x)dx} = \int (Q(x) \cdot e^{\int P(x)dx}) dx + C$  which is equivalent to  $y = \frac{1}{e^{\int P(x)dx}} \left( \int (Q(x) \cdot e^{\int P(x)dx}) dx + C \right)$ ".

Students should now be able to solve a first order linear differential equation. You will probably want to complete this example as a class to be sure that all students understand the process.

The screenshot shows the text: "Now it is our turn to try to solve a first order differential equation. Let's work this problem together. Solve:  $xy' + 3y = x^3$ ".

To solve this differential equation, you will first need to rewrite it into the standard form.

$$xy' + 3y = x^3$$

$$\Rightarrow y' + \frac{3}{x}y = x^2$$

Students should now identify  $P(x) = \frac{3}{x}$  and  $Q(x) = x^2$ .

Calculating the value of the integrating factor,

$$u(x) = e^{\int P(x)dx}$$

$$u(x) = e^{\int \frac{3}{x}dx}$$

$$u(x) = e^{3 \ln x}$$

$$u(x) = x^3$$

Therefore the general solution to the differential equation is

$$y = \frac{1}{u(x)} \left( \int Q(x)u(x)dx + C \right)$$

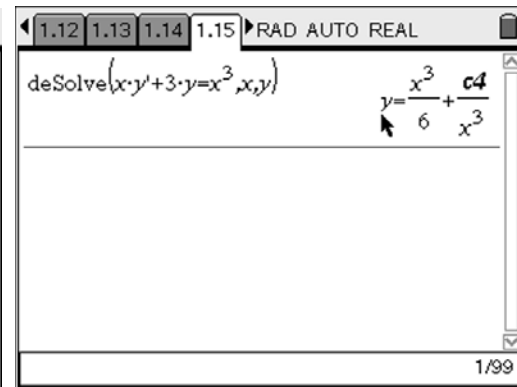
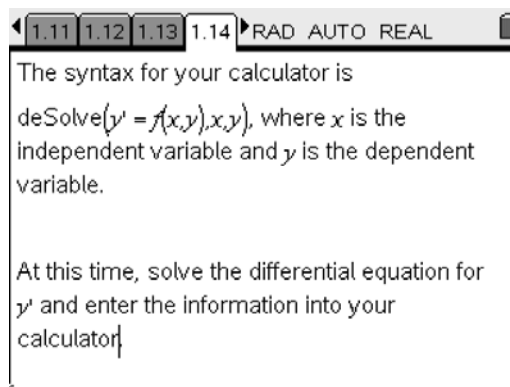
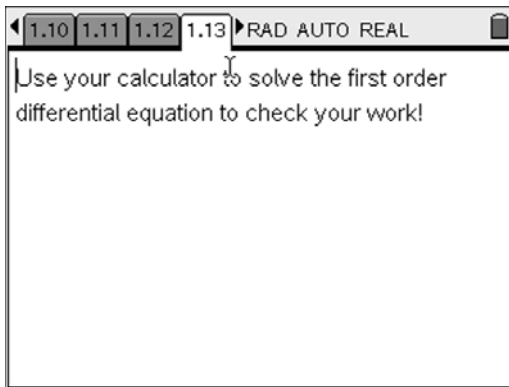
$$y = \frac{1}{x^3} \left( \int (x^2 \cdot x^3)dx + C \right)$$

$$y = \frac{1}{x^3} \left( \int (x^5)dx + C \right)$$

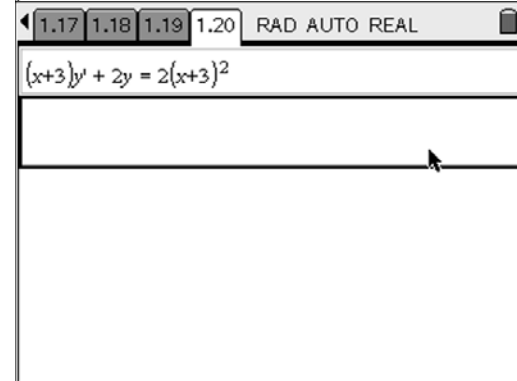
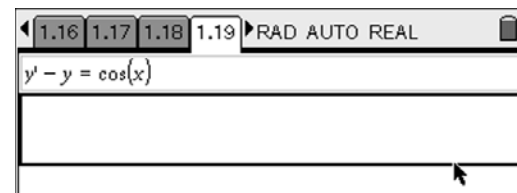
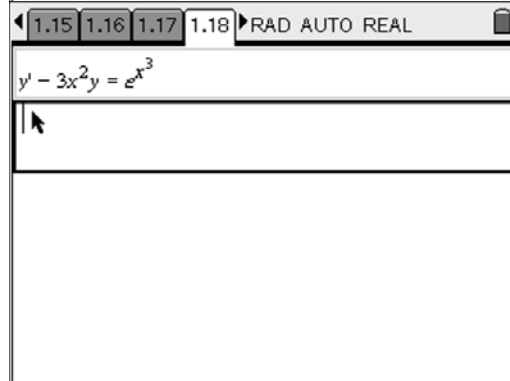
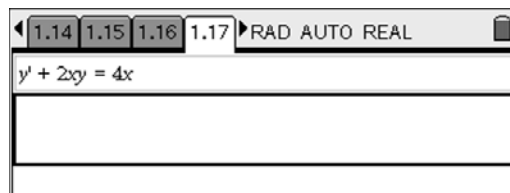
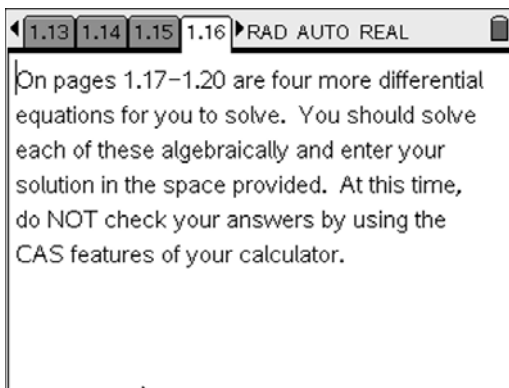
$$y = \frac{1}{x^3} \cdot \frac{x^6}{6} + \frac{C}{x^3}$$

$$y = \frac{1}{6}x^3 + \frac{C}{x^3}$$

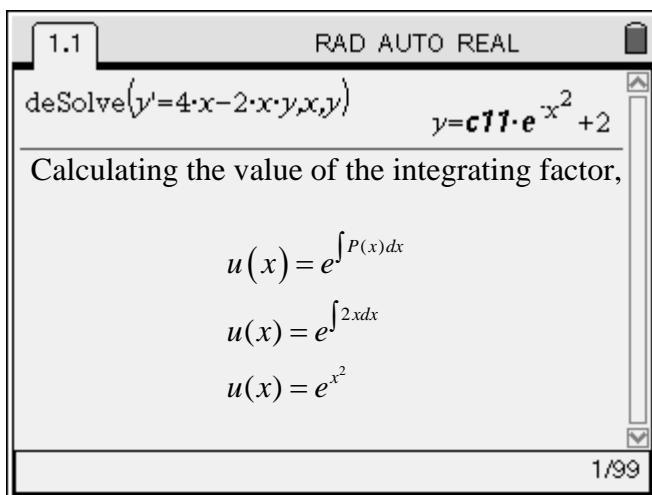
At this point, you should show students how to solve a first order linear differential equation with their Nspire-CAS calculators. The procedure can be found on pages 1.13 – 1.15.



Students should now attempt to solve the four differential equation on pages 1.17 – 1.20.



The solutions to the differential equations appear below.



$$P(x) = 2x \text{ and } Q(x) = 4x$$

$$y = \frac{1}{u(x)} \left( \int Q(x)u(x)dx + C \right)$$

$$y = \frac{1}{e^{x^2}} \left( \int (4xe^{x^2}) dx + C \right)$$

$$y = \frac{4}{e^{x^2}} \left( \int (xe^{x^2}) dx + C \right) \quad \begin{matrix} u = x^2 \\ du = 2xdx \end{matrix}$$

$$y = \frac{2}{e^{x^2}} \left( \int (e^u) du + C \right)$$

$$y = \frac{2}{e^{x^2}} \left( e^{x^2} + C \right)$$

$$y = 2 + \frac{C}{e^{x^2}}$$

1.1 1.2 RAD AUTO REAL

deSolve(y'=cos(x)+y,x,y)

$$y = \frac{-\cos(x)}{2} + \frac{\sin(x)}{2} + c12 \cdot e^{-x}$$


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Calculating the value of the integrating factor,

$$u(x) = e^{\int P(x) dx}$$

$$u(x) = e^{-\int dx}$$

$$u(x) = e^{-x}$$

1/99

$$P(x) = -1 \text{ and } Q(x) = \cos x$$

$$y = \frac{1}{u(x)} \left( \int Q(x)u(x) dx + C \right)$$

$$y = \frac{1}{e^{-x}} \left( \int (e^{-x} \cos x) dx + C \right)$$

$$y = \frac{1}{e^{-x}} \left( \frac{1}{2} e^{-x} (\sin x - \cos x) + C \right)$$

$$y = \frac{1}{2} (\sin x - \cos x) + Ce^x$$

Use  
integration by  
parts.

1.1 1.2 1.3 1.4 RAD AUTO REAL

deSolve(y'=e^{x^3}+3\*x^2\*y,x,y) y=(x+c15)\*e^{-x^3}

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Calculating the value of the integrating factor,

$$u(x) = e^{\int P(x) dx}$$

$$u(x) = e^{\int -3x^2 dx}$$

$$u(x) = e^{-x^3}$$

1/99

$$P(x) = -3x^2 \text{ and } Q(x) = e^{x^3}$$

$$y = \frac{1}{u(x)} \left( \int Q(x)u(x) dx + C \right)$$

$$y = \frac{1}{e^{-x^3}} \left( \int (e^{x^3} \cdot e^{-x^3}) dx + C \right)$$

$$y = \frac{1}{e^{-x^3}} \left( \int dx + C \right)$$

$$y = \frac{1}{e^{-x^3}} (x + C)$$

1.1 1.2 1.3 1.4 RAD AUTO REAL

deSolve(y'=\frac{2\*(x+3)^2-2\*y}{x+3},x,y)

$$y = \frac{x^4 + 12x^3 + 54x^2 + 108x + 2 \cdot c14 + 81}{2 \cdot (x+3)^2}$$


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Calculating the value of the integrating factor,

$$u(x) = e^{\int P(x) dx}$$

1/99

$$(x+3)y' + 2y = 2(x+3)^2$$

$$y' + \frac{2}{x+3}y = 2(x+3)$$

$$P(x) = \frac{2}{x+3} \text{ and } Q(x) = 2(x+3)$$

$$y = \frac{1}{u(x)} \left( \int Q(x)u(x) dx + C \right)$$

$$y = \frac{1}{(x+3)^2} \left( \int 2(x+3) \cdot (x+3)^2 dx + C \right)$$

$$y = \frac{2}{(x+3)^2} \left( \int (x+3)^3 dx + C \right)$$

$$y = \frac{1}{2(x+3)^2} \left( (x+3)^4 + C \right)$$

$$y = \frac{(x+3)^2}{2} + \frac{C}{(x+3)^2}$$

$$u(x) = e^{2 \int \frac{dx}{x+3}}$$

$$u(x) = e^{2 \ln|x+3|}$$

$$u(x) = (x+3)^2$$

After the students have solved each of their differential equations algebraically, collect their solutions using the TI-Nspire Navigator system.

As you are showing the results to the four questions to the class, using the slide show within Class Analysis, discuss the different format that the answers have been expressed. It is important to point out to the students that answers may be equivalent and correct but look different.

While reviewing the results, you should have the students solve each of these differential equations using their TI-Nspire CAS handheld. Using the TI-Nspire Navigator system, select four different students to present their solutions to one of the four differential equations. Discuss the solution the calculator gave versus the one that was derived algebraically from the students.