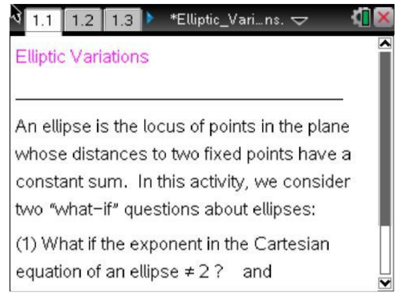
Elliptic Variations

Student Activity

Name ________________

Class ________________

Open the TI-Nspire document *Elliptic_Variations.tns.*



A standard ellipse has the Cartesian equation $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ and is the locus of points in the plane whose distances to two fixed points have a constant sum. In this activity, we will consider two "what-if" questions:
(1) What if the exponent in the Cartesian equation ≠ 2?
(2) What if "sum" is changed to "product"?

Problem 1: Superellipses

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Press ctrl ▶ and ctrl ◀ to navigate through the lesson.

The curves described by the equation $\left|\frac{x}{a}\right|^n+\left|\frac{y}{b}\right|^n=1$, where n is a positive rational number, are called **superellipses**.

To conveniently explore fractional values of n, we set $n = \frac{k}{6}$ and consider the values of n from $\frac{1}{6}$ to 4, or equivalently, the values of k from 1 to 24.

Piet Hein popularized these curves around 1960 for values of n > 2 [especially n = 2.5]. The curves for these values of n are "rounded rectangles" that have been used in the design of a town "square" in Sweden to improve traffic flow and in the design of tabletops and other furniture.

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We will use parametric equations to produce a graph of these superellipses. The derivation of these equations is shown later in this section.

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Use the a-clicker and b-clicker to set the values of a and b to a = 6 and b = 4 Using the k-clicker, scroll through the values of k = 1 to 24. As you answer the questions below, you might want to experiment with other values of a and b to confirm your responses.

1. Which value of k generates the graph of a standard ellipse?

2. Which value of k generates the graph of a polygon, instead of a curve? Identify the type of polygon.

3. Let k* denote the value of k in your answer to Question #2. Describe the graphs for 1 ≤ k ≤ k*.

4. Use the a-clicker and b-clicker to set the values of a and b to a = 6 and b = 6 Using the k-clicker, scroll through the values of k = 1 to 24. Describe the curve when $a = b$.

5. What do you predict will happen to the graph of $\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1$ as n becomes very large? Why?

6. What do you predict will happen to the graph of $\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1$ as n becomes very small? Why?

Recall that the parametric equations for the standard ellipse are $x = a \cdot \cos(t)$ and $y = b \cdot \sin(t)$ for $0 \le t \le 2\pi$.

To modify these equations for a superellipse, we have $\begin{cases} x = \pm a \cdot |\cos(t)|^{2/n} \\ y = \pm b \cdot |\sin(t)|^{2/n} \end{cases}$.

These equations are valid because $\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = \left(|\cos(t)|^{2/n}\right)^n + \left(|\sin(t)|^{2/n}\right)^n$
$= (\cos(t))^2 + (\sin(t))^2 = 1$.

The difficulty with using these equations is that there are four of them. We would need to draw four different curves corresponding to the four possible combinations of + (plus) and - (minus). Fortunately, we can incorporate the **signum** function, $sign(x)$, into these equations to combine all of them into one equation. The signum function is defined by

$$sign(x) = \begin{cases} 1 & for\ x > 0 \\ 0 & for\ x = 0 \\ -1 & for\ x < 0 \end{cases}$$

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Note: On the calculator, $sign(0) \neq 0$. This anomaly does not affect the graphs in this problem.

The parametric equations for a superellipse become

$$\begin{cases} x = a \cdot |\cos(t)|^{2/n} \cdot sign(\cos(t)) \\ y = b \cdot |\sin(t)|^{2/n} \cdot sign(\sin(t)) \end{cases} \quad \text{for } 0 \le t \le 2\pi.$$

7. Explain carefully how these equations determine the set of points on the graph in the second quadrant when $\frac{\pi}{2} \le x \le \pi$.

Problem 2: Cassini Ovals

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A **Cassini oval** is the locus of points in the plane whose distances to two fixed points have a constant product. Giovanni Domenico Cassini developed this set of curves in 1680 because he believed that the motion of the Earth and Sun followed the path of a Cassini oval.

In other words, a Cassini oval is the graph of the equation $\sqrt{(x-a)^2 + y^2} \cdot \sqrt{(x+a)^2 + y^2} = b^2$ for various values of a and b. The fixed points are ($-a$,0) and (a,0), and the distance is b^2. In this activity, we consider the case when a = 4.

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We will use polar equations to produce graphs of Cassini ovals. The derivation of these equations is shown later in this section.

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Set the value of the b-clicker to b = 1.6. Scroll through the values for b = 1.6 down to b = 1.0.

8. Describe how the curve changes for values of b between 1.6 and 1.0.

9. What do you predict will happen to the curve as b becomes very large? Why?

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Set the value of the c-clicker to c = 0.95. Scroll through the values for c = 0.95 down to c = 0.5.

Note: Because of an anomaly in the way the calculator draws such graphs, the graphs of the ovals appear to be open, or disconnected, but, in fact, they are closed, connected graphs.

10. Describe how the curve changes for values of c between 0.95 and 0.5.

11. What do you predict will happen to the curve as c becomes very small?

To derive the general polar equations of $\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2}=b^2$, we start by letting $x=r\cos\theta$ and $y=r\sin\theta$. Then

(1) $[(r\cos\theta-a)^2+(r\sin\theta-a)^2]\cdot[(r\cos\theta+a)^2+(r\sin\theta+a)^2]=b^4$

(2) $r^4[\cos^4\theta+\sin^4\theta+2\sin^2\theta\cos^2\theta]-2a^2r^2[\cos^2\theta-\sin^2\theta]+a^4=b^4$ =

(3) $r^4-2a^2r^2\cos 2\theta+a^4=b^4$

Solving for r^2 and then r gives

(4) $r^2=a^2\cos 2\theta\pm\sqrt{a^4\cos^2 2\theta-a^4+b^4}$ or

(5) $r^2=a^2\left(\cos 2\theta\pm\sqrt{\left(\frac{b}{a}\right)^4-\sin^2 2\theta}\right)$ so that

(6) $r=\pm a\sqrt{\cos 2\theta\pm\sqrt{\left(\frac{b}{a}\right)^4-\sin^2 2\theta}}$.

12. Explain how to go from step (2) to step (3) and from step (4) to step (5) in the argument above.