TI-nspire™

Grade level: 9-12
Subject: mathematics
Time required: 45 minutes

The Lunes of Hippocrates

by Karen Droga Campe

Activity overview

In this activity, students will explore a figure that involves lunes – the area enclosed between arcs of intersecting circles. When lunes are constructed on the sides of a right triangle, an interesting result occurs.

Concepts

- *Right triangles and the Pythagorean Theorem*
- *Area of circles*

Teacher preparation

- *Circles are constructed on each side of the right triangle ABC, and the areas of the semicircles are calculated.*
- *In the constructed figure, the lunes are defined as the regions inside the two small semicircles (with diameters* $\overline{AC}$ *and* $\overline{BC}$*, but outside the large semicircle (with diameter* $\overline{AB}$*). The sum of the areas of the lunes can be calculated by adding the areas of the two smaller semicircles to the area of the triangle and subtracting the area of the large semicircle.*
- *Depending on student skill level, students can construct the figure themselves (Problem 1) or move directly to the pre-made figure (Problem 2).*
- *Problem 3 is an extension that uses algebra to prove the conjecture from Problem 2.*

Classroom management tips

- *This activity is designed to be* ***student-centered*** *with the teacher acting as a facilitator while students work cooperatively. Use the following pages as a framework as to how the activity will progress.*
- *The student worksheet provides a place for students to record their answers and observations.*
- *The Document Settings for the TI-Nspire can be accessed by pressing* (home on) (5) (2) (2) *(**Home > Settings & Status > Settings > Graphs & Geometry**). Geometry Angle should be set to "Degree" and the desired Display Digits can be set as well.*

TI-Nspire Applications

Graphs & Geometry, Lists & Spreadsheet, Notes

by: Karen Droga Campe
Grade level: secondary
Subject: mathematics
Time required: 45 minutes

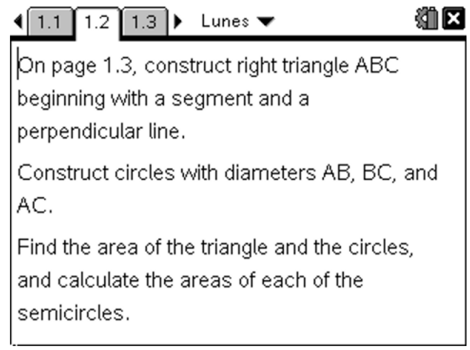
Materials: TI-Nspire/TI-Nspire CAS

Problem 1 – Constructing the Lunes

Step 1: Have students open the file Lunes. Read the directions on page 1.2.

A lune is the area enclosed by arcs of intersecting circles.

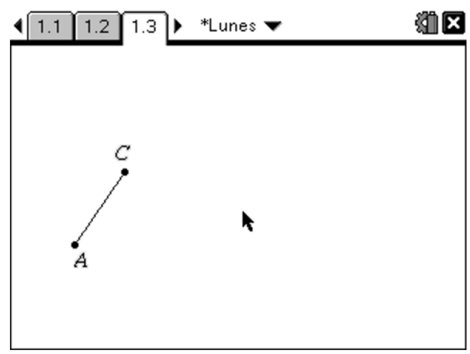
The circles will be created on the sides of a right triangle.



Step 2: On page 1.3, select the **Segment** tool and construct a segment with endpoints A and C.

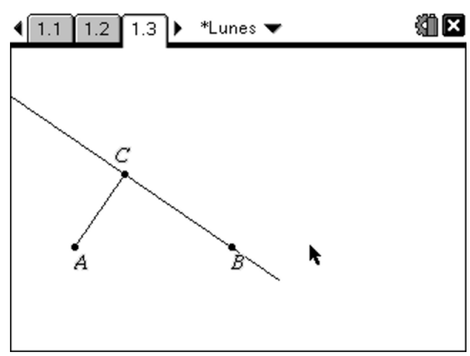
Press shift A immediately after creating the first endpoint, and press shift C immediately after creating the second endpoint.

Note: If the endpoints were not labeled as they were created, use the **Text** tool to label them.

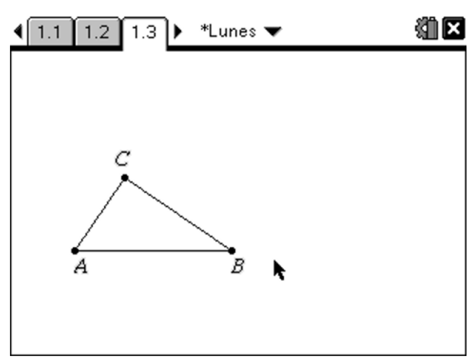


Step 3: Select the **Perpendicular** tool and construct a line through point C perpendicular to $\overline{AC}$.

Select the **Point On** tool and create point B on the perpendicular line.

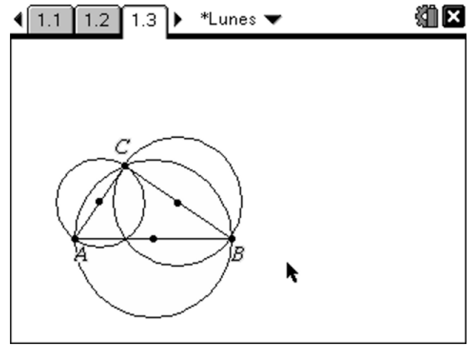


Step 4: Use the **Triangle** tool to create triangle ABC. Then select the **Hide/Show** tool and hide the perpendicular line.



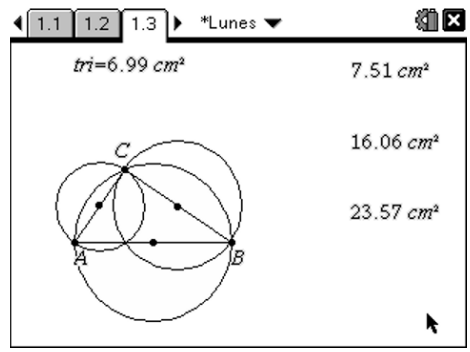
Step 5: Create the midpoints of the sides of ΔABC with the **Midpoint** tool.

Use the midpoints as centers to create three circles with the **Circle** tool. Each circle will have a side of ΔABC as its diameter.



Step 6: Select the **Measure >Area** tool and measure the areas of the three circles and the triangle.

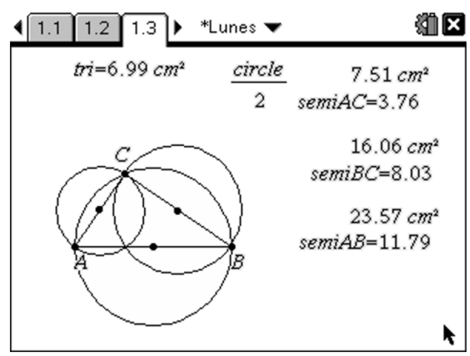
Note: Press (enter) first to select the shape, then press (enter) to anchor the measurement on the page.



Step 7: Use the **Text** tool to put the expression $\frac{circle}{2}$ on the screen as shown.

Note: Press (enter) to begin the text box and press (enter) to end the text box.

Use the **Calculate** tool to calculate the area of each semicircle using this expression.



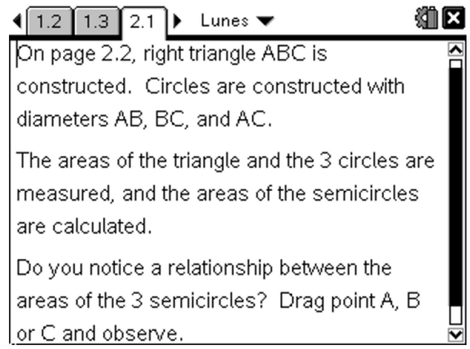
Step 8: Do you notice a relationship between the areas of the 3 semicircles? Drag point A, B or C and observe.

Enter the areas of the 3 semicircles on the worksheet #6 and make a conjecture.

What well-known theorem justifies this result?

Problem 2 – Sum of the Areas of the Lunes

Step 1: Advance to page 2.1 and read the directions.

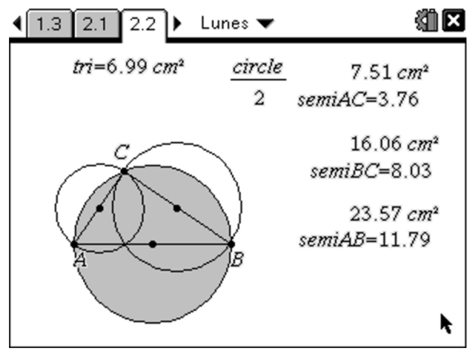


Step 2: Advance to page 2.2. If you skipped problem 1, answer these questions now:

Do you notice a relationship between the areas of the 3 semicircles? Drag point A, B or C and observe.

Enter the areas of the 3 semicircles on the worksheet #6 and make a conjecture.

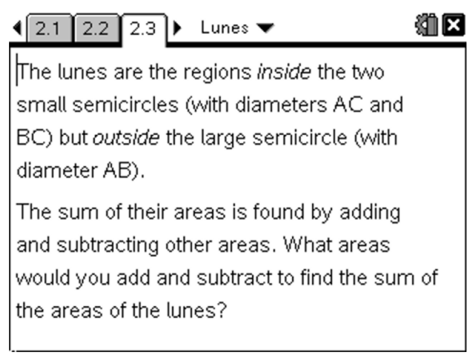
What well-known theorem justifies this result?



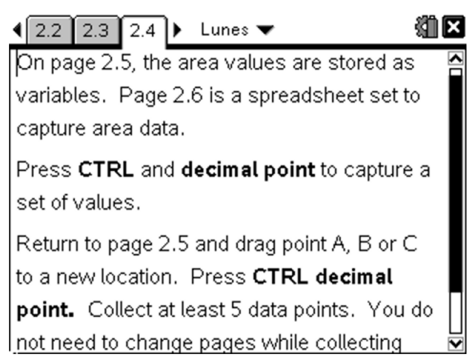
Step 3: On page 2.3, the lunes are defined as the regions *inside* the two small semicircles (with diameters $\overline{AC}$ and $\overline{BC}$, but *outside* the large semicircle (with diameter $\overline{AB}$).

What areas would you add and subtract to find the sum of the areas of the lunes?

Write a formula on the worksheet #9.



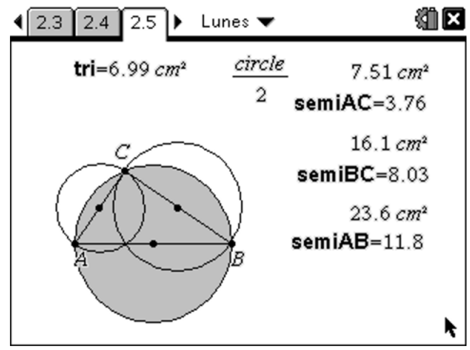
Step 4: Read the directions on page 2.4.

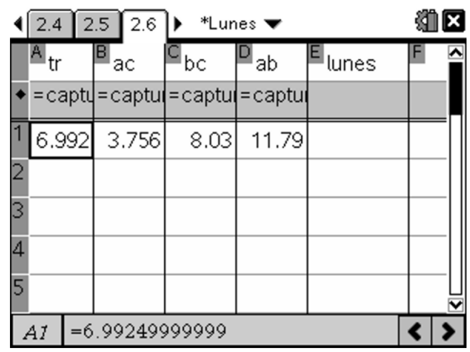


Step 5: On page 2.5, the area values are stored as variables. Page 2.6 is a spreadsheet set to capture area data.

Drag any of the points A, B, or C to new locations. Press ctrl . to capture the values of the variables.

Advance to page 2.6 and the first set of values is stored in the spreadsheet.



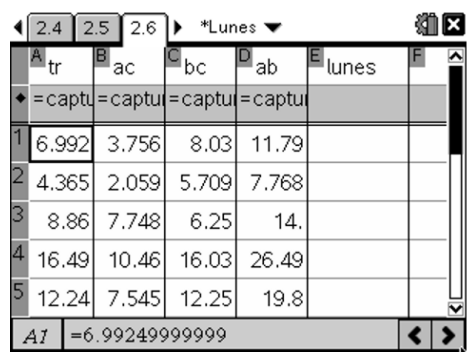


Step 6: Return to page 2.5 and drag at least two of the points A, B, or C again.

Again press ctrl . to capture the data.

Repeat this 3 more times so you have captured 5 sets of data. There is no need to change pages while capturing data.

Record the data on the worksheet #10.

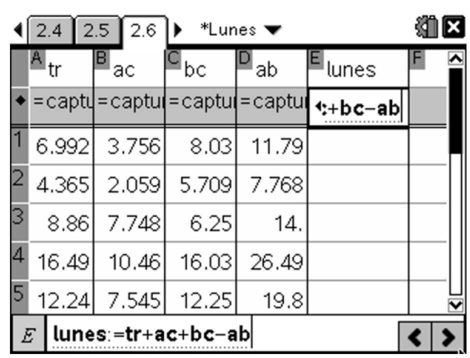


Step 7: Advance to page 2.6 and examine the data.

Move the cursor to the formula entry row (marked with ♦) for column E, which is named lunes.

Press = and enter the formula
tr + ac + bc – ab.

Press enter to complete the formula. The sum of the areas of the lunes are calculated in column E.

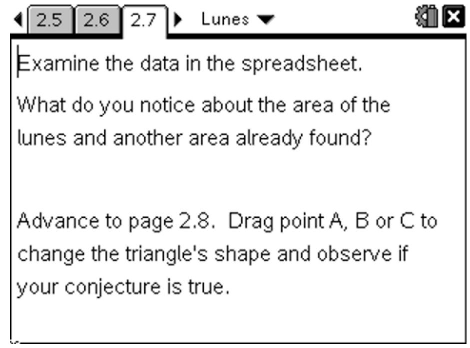


Step 8: Read page 2.7.

What do you notice about the area of the lunes and another area already found?

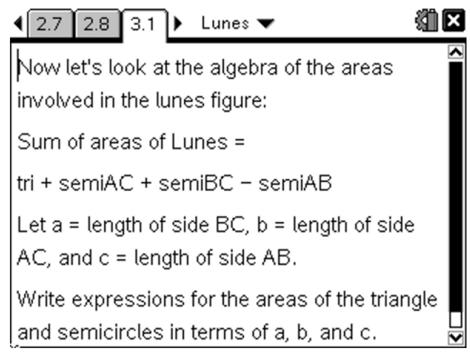
Record your conjecture on the worksheet #12.

Advance to page 2.8. Drag point A, B or C to change the triangle's shape and observe if your conjecture is true.



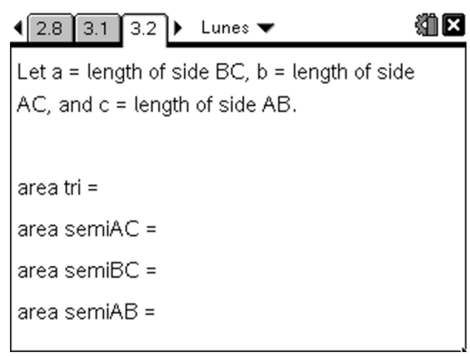
Problem 3 – Algebraic Approach (Extension)

Step 1: Read page 3.1.



Step 2: Let a = length of side $\overline{BC}$, b = length of side $\overline{AC}$, and c = length of side $\overline{AB}$.

On the worksheet #14, write expressions for the areas of the triangle and semicircles in terms of a, b, and c.

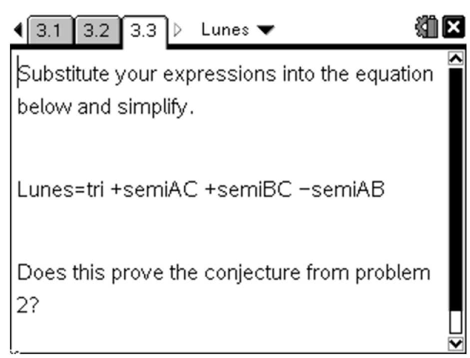


Step 3: In #15 on the worksheet, substitute your expressions into this equation and simplify.

Lunes = triangle + semiAC + semiBC – semiAB

Does this prove the conjecture from problem 2?

If desired, use the CAS capabilities of TI-Nspire to complete the algebra.



Assessment and evaluation

- *In Problem 1, students are to construct the figure and measure the areas of the triangle and circles. After calculating the areas of the semicircles, they should drag a vertex of the triangle and observe the areas as they change. The Pythagorean Theorem holds true for the areas of the 3 semicircles (the sum of the areas of the semicircles on the legs $\overline{AC}$ and $\overline{BC}$ equals the area of the semicircle on the hypotenuse $\overline{AB}$). If Problem 1 is skipped, these same questions are posed on page 2.1 of the tns document.*

- *In Problem 2, students are calculating the sum of the areas of the lunes using the formula*
  LUNES = TRIANGLE + semiAC + semiBC – semiAB
  Students collect data as they change the shape of $\triangle ABC$. Be sure that students drag at least two vertices each time they collect a new data point. The calculations in the spreadsheet will support the conjecture that the sum of the areas of the lunes equals the area of the triangle.

Activity extensions

- *Problem 3 is an Extension that proves the conjecture. If the sides of the triangle are a, b, and c, the areas of the semicircles can be expressed as $\pi\left(\frac{a}{2}\right)^2$, $\pi\left(\frac{b}{2}\right)^2$, and $\pi\left(\frac{c}{2}\right)^2$. When these expressions are substituted into the formula above and simplified, the conjecture can be proven with the Pythagorean Theorem.*

- *Hippocrates was a Greek mathematician who examined the lune problem. His figure was a square inscribed in a circle, with another semicircle constructed on one side of the square. Create this figure, and show that the area of the lune on one side of the square is equal to the area of the isosceles right triangle that is one-quarter of the square.*

References

Nelson, David (Ed.), Penguin Dictionary of Mathematics, Second Edition, 1998.

Wells, David, The Penguin Dictionary of Curious and Interesting Geometry, 1991.

Student TI-Nspire Document *Lunes.tns.*