

## The Lunes of Hippocrates

by Karen Droga Campe

### Activity overview

*In this activity, students will explore a figure that involves lunes – the area enclosed between arcs of intersecting circles. When lunes are constructed on the sides of a right triangle, an interesting result occurs.*

### Concepts

- *Right triangles and the Pythagorean Theorem*
- *Area of circles*

### Teacher preparation

- *Circles are constructed on each side of the right triangle ABC, and the areas of the semicircles are calculated.*
- *In the constructed figure, the lunes are defined as the regions inside the two small semicircles (with diameters  $\overline{AC}$  and  $\overline{BC}$ , but outside the large semicircle (with diameter  $\overline{AB}$ ). The sum of the areas of the lunes can be calculated by adding the areas of the two smaller semicircles to the area of the triangle and subtracting the area of the large semicircle.*
- *Depending on student skill level, students can construct the figure themselves (Problem 1) or move directly to the pre-made figure (Problem 2).*
- *Problem 3 is an extension that uses algebra to prove the conjecture from Problem 2.*

### Classroom management tips

- *This activity is designed to be **student-centered** with the teacher acting as a facilitator while students work cooperatively. Use the following pages as a framework as to how the activity will progress.*
- *The student worksheet provides a place for students to record their answers and observations.*
- *The Document Settings for the TI-Nspire can be accessed by pressing  $\left(\frac{\text{on}}$  5 2 2) (**Home > Settings & Status > Settings > Graphs & Geometry**). Geometry Angle should be set to “Degree” and the desired Display Digits can be set as well.*

### TI-Nspire Applications

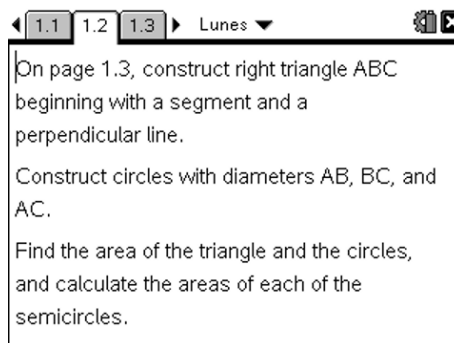
*Graphs & Geometry, Lists & Spreadsheet, Notes*

**Problem 1 – Constructing the Lunes**

**Step 1:** Have students open the file Lunes. Read the directions on page 1.2.

A lune is the area enclosed by arcs of intersecting circles.

The circles will be created on the sides of a right triangle.



On page 1.3, construct right triangle ABC beginning with a segment and a perpendicular line.

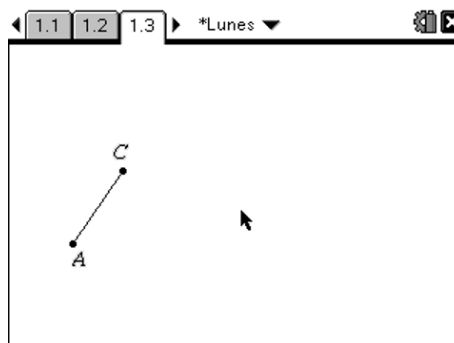
Construct circles with diameters AB, BC, and AC.

Find the area of the triangle and the circles, and calculate the areas of each of the semicircles.

**Step 2:** On page 1.3, select the **Segment** tool and construct a segment with endpoints A and C.

Press  $\text{[shift] [A]}$  immediately after creating the first endpoint, and press  $\text{[shift] [C]}$  immediately after creating the second endpoint.

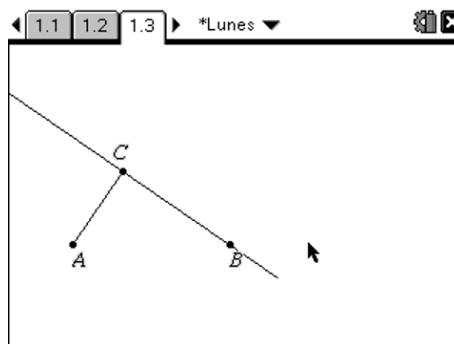
**Note:** If the endpoints were not labeled as they were created, use the **Text** tool to label them.



A screenshot of the TI-Nspire interface showing a segment with endpoints labeled A and C. The interface includes page navigation buttons (1.1, 1.2, 1.3) and a window title '\*Lunes'.

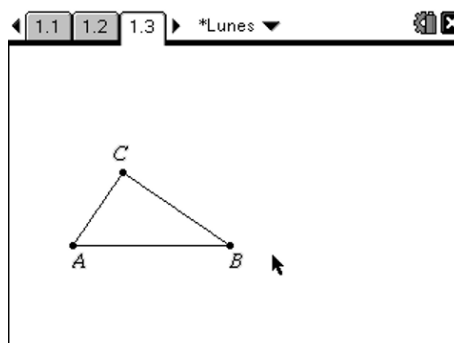
**Step 3:** Select the **Perpendicular** tool and construct a line through point C perpendicular to  $\overline{AC}$ .

Select the **Point On** tool and create point B on the perpendicular line.



A screenshot of the TI-Nspire interface showing a line passing through point C, perpendicular to segment AC. Point B is located on this perpendicular line. The interface includes page navigation buttons (1.1, 1.2, 1.3) and a window title '\*Lunes'.

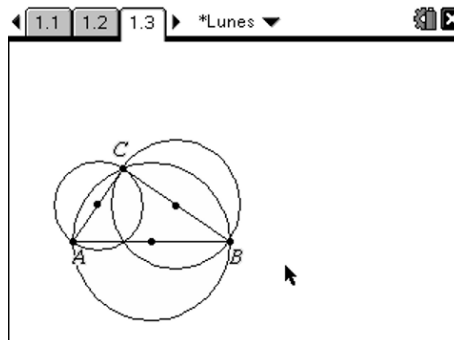
**Step 4:** Use the **Triangle** tool to create triangle ABC. Then select the **Hide/Show** tool and hide the perpendicular line.



A screenshot of the TI-Nspire interface showing the final right triangle ABC. The perpendicular line from step 3 is hidden. The interface includes page navigation buttons (1.1, 1.2, 1.3) and a window title '\*Lunes'.

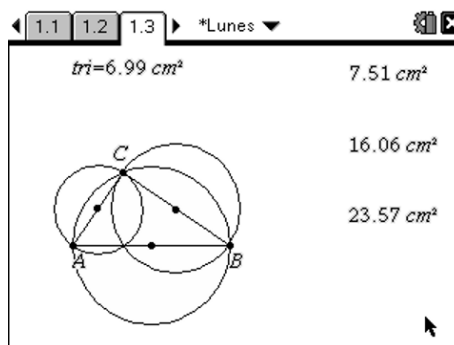
**Step 5:** Create the midpoints of the sides of  $\triangle ABC$  with the **Midpoint** tool.

Use the midpoints as centers to create three circles with the **Circle** tool. Each circle will have a side of  $\triangle ABC$  as its diameter.



**Step 6:** Select the **Measure > Area** tool and measure the areas of the three circles and the triangle.

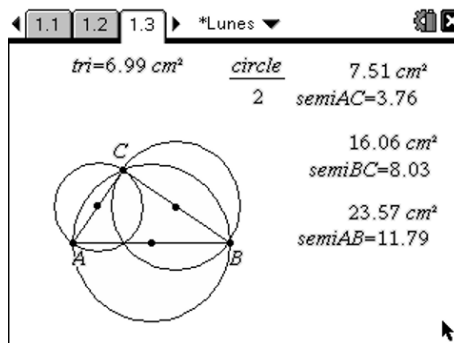
**Note:** Press **(enter)** first to select the shape, then press **(enter)** to anchor the measurement on the page.



**Step 7:** Use the **Text** tool to put the expression  $\frac{\text{circle}}{2}$  on the screen as shown.

**Note:** Press **(enter)** to begin the text box and press **(enter)** to end the text box.

Use the **Calculate** tool to calculate the area of each semicircle using this expression.



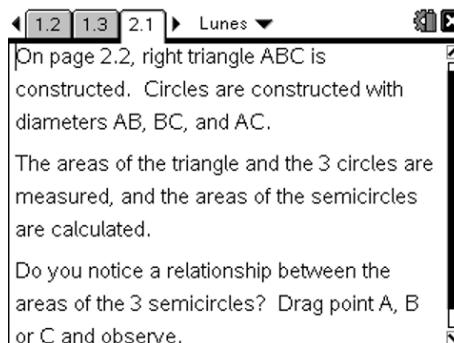
**Step 8:** Do you notice a relationship between the areas of the 3 semicircles? Drag point A, B or C and observe.

Enter the areas of the 3 semicircles on the worksheet #6 and make a conjecture.

What well-known theorem justifies this result?

**Problem 2 – Sum of the Areas of the Lunes**

**Step 1:** Advance to page 2.1 and read the directions.



1.2 1.3 2.1 Lunes

On page 2.2, right triangle ABC is constructed. Circles are constructed with diameters AB, BC, and AC.

The areas of the triangle and the 3 circles are measured, and the areas of the semicircles are calculated.

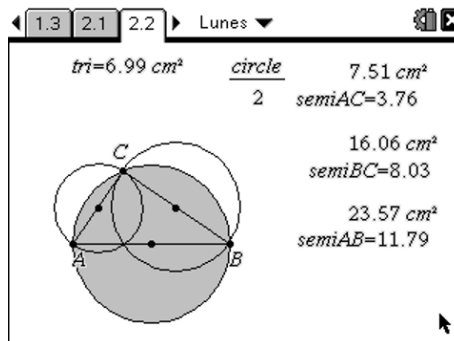
Do you notice a relationship between the areas of the 3 semicircles? Drag point A, B or C and observe.

**Step 2:** Advance to page 2.2. If you skipped problem 1, answer these questions now:

Do you notice a relationship between the areas of the 3 semicircles? Drag point A, B or C and observe.

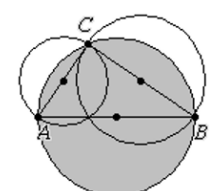
Enter the areas of the 3 semicircles on the worksheet #6 and make a conjecture.

What well-known theorem justifies this result?



1.3 2.1 2.2 Lunes

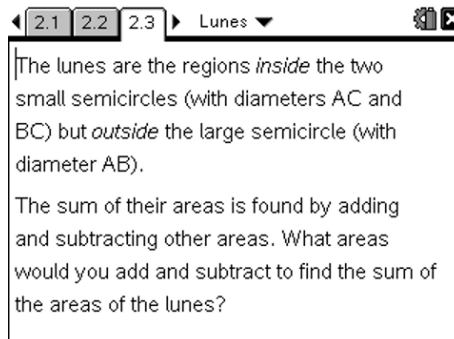
$tri = 6.99 \text{ cm}^2$	$\frac{\text{circle}}{2}$	$7.51 \text{ cm}^2$
		$semiAC = 3.76$
		$16.06 \text{ cm}^2$
		$semiBC = 8.03$
		$23.57 \text{ cm}^2$
		$semiAB = 11.79$



**Step 3:** On page 2.3, the lunes are defined as the regions *inside* the two small semicircles (with diameters  $\overline{AC}$  and  $\overline{BC}$ ), but *outside* the large semicircle (with diameter  $\overline{AB}$ ).

What areas would you add and subtract to find the sum of the areas of the lunes?

Write a formula on the worksheet #9.

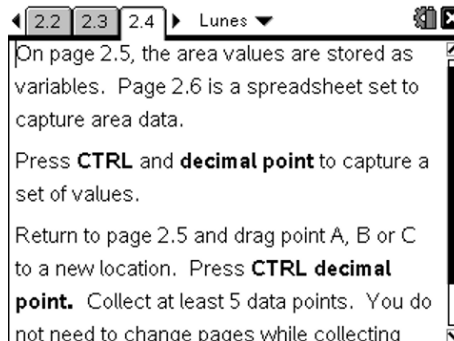


2.1 2.2 2.3 Lunes

The lunes are the regions *inside* the two small semicircles (with diameters AC and BC) but *outside* the large semicircle (with diameter AB).

The sum of their areas is found by adding and subtracting other areas. What areas would you add and subtract to find the sum of the areas of the lunes?

**Step 4:** Read the directions on page 2.4.



2.2 2.3 2.4 Lunes

On page 2.5, the area values are stored as variables. Page 2.6 is a spreadsheet set to capture area data.

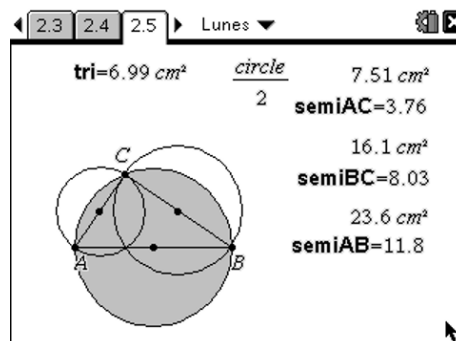
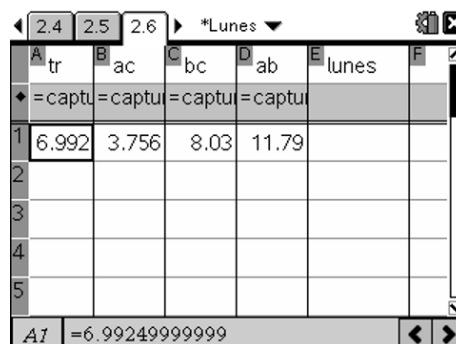
Press **CTRL** and **decimal point** to capture a set of values.

Return to page 2.5 and drag point A, B or C to a new location. Press **CTRL decimal point**. Collect at least 5 data points. You do not need to change pages while collecting

**Step 5:** On page 2.5, the area values are stored as variables. Page 2.6 is a spreadsheet set to capture area data.

Drag any of the points A, B, or C to new locations. Press  $\text{(ctrl)} \text{(.)}$  to capture the values of the variables.

Advance to page 2.6 and the first set of values is stored in the spreadsheet.

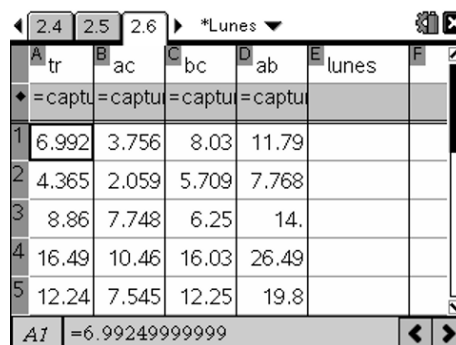
	A tr	B ac	C bc	D ab	E lunes
◆	=captu	=captu	=captu	=captu	
1	6.992	3.756	8.03	11.79	
2					
3					
4					
5					

**Step 6:** Return to page 2.5 and drag at least two of the points A, B, or C again.

Again press  $\text{(ctrl)} \text{(.)}$  to capture the data.

Repeat this 3 more times so you have captured 5 sets of data. There is no need to change pages while capturing data.

Record the data on the worksheet #10.



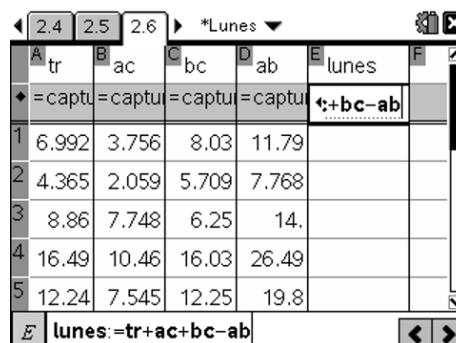
	A tr	B ac	C bc	D ab	E lunes
◆	=captu	=captu	=captu	=captu	
1	6.992	3.756	8.03	11.79	
2	4.365	2.059	5.709	7.768	
3	8.86	7.748	6.25	14.	
4	16.49	10.46	16.03	26.49	
5	12.24	7.545	12.25	19.8	

**Step 7:** Advance to page 2.6 and examine the data.

Move the cursor to the formula entry row (marked with ◆) for column E, which is named lunes.

Press  $\text{(=)}$  and enter the formula **tr + ac + bc - ab.**

Press  $\text{(enter)}$  to complete the formula. The sum of the areas of the lunes are calculated in column E.



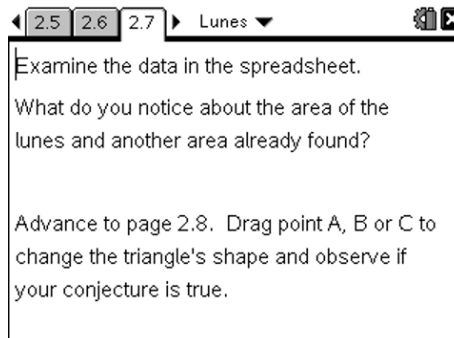
	A tr	B ac	C bc	D ab	E lunes
◆	=captu	=captu	=captu	=captu	←tr+ac+bc-ab
1	6.992	3.756	8.03	11.79	
2	4.365	2.059	5.709	7.768	
3	8.86	7.748	6.25	14.	
4	16.49	10.46	16.03	26.49	
5	12.24	7.545	12.25	19.8	

**Step 8:** Read page 2.7.

What do you notice about the area of the lunes and another area already found?

Record your conjecture on the worksheet #12.

Advance to page 2.8. Drag point A, B or C to change the triangle's shape and observe if your conjecture is true.



2.5 2.6 2.7 Lunes

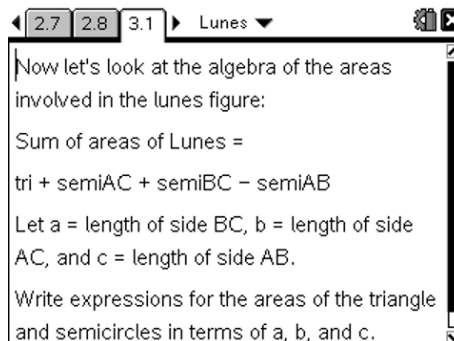
Examine the data in the spreadsheet.

What do you notice about the area of the lunes and another area already found?

Advance to page 2.8. Drag point A, B or C to change the triangle's shape and observe if your conjecture is true.

### Problem 3 – Algebraic Approach (Extension)

**Step 1:** Read page 3.1.



2.7 2.8 3.1 Lunes

Now let's look at the algebra of the areas involved in the lunes figure:

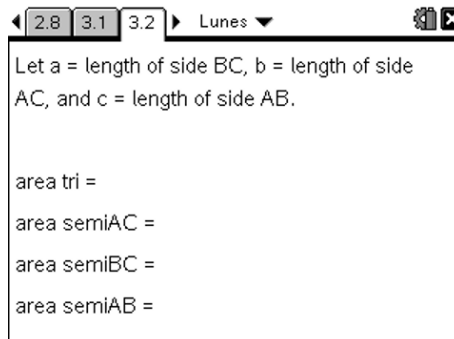
Sum of areas of Lunes =  
 $\text{tri} + \text{semiAC} + \text{semiBC} - \text{semiAB}$

Let a = length of side BC, b = length of side AC, and c = length of side AB.

Write expressions for the areas of the triangle and semicircles in terms of a, b, and c.

**Step 2:** Let a = length of side  $\overline{BC}$ , b = length of side  $\overline{AC}$ , and c = length of side  $\overline{AB}$ .

On the worksheet #14, write expressions for the areas of the triangle and semicircles in terms of a, b, and c.



2.8 3.1 3.2 Lunes

Let a = length of side BC, b = length of side AC, and c = length of side AB.

area tri =

area semiAC =

area semiBC =

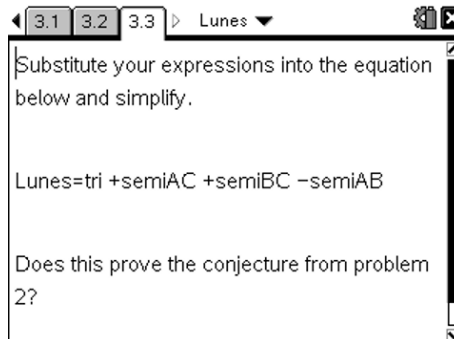
area semiAB =

**Step 3:** In #15 on the worksheet, substitute your expressions into this equation and simplify.

$$\text{Lunes} = \text{triangle} + \text{semiAC} + \text{semiBC} - \text{semiAB}$$

Does this prove the conjecture from problem 2?

If desired, use the CAS capabilities of TI-Nspire to complete the algebra.



3.1 3.2 3.3 Lunes

Substitute your expressions into the equation below and simplify.

$\text{Lunes} = \text{tri} + \text{semiAC} + \text{semiBC} - \text{semiAB}$

Does this prove the conjecture from problem 2?

**Assessment and evaluation**

- *In Problem 1, students are to construct the figure and measure the areas of the triangle and circles. After calculating the areas of the semicircles, they should drag a vertex of the triangle and observe the areas as they change. The Pythagorean Theorem holds true for the areas of the 3 semicircles (the sum of the areas of the semicircles on the legs  $\overline{AC}$  and  $\overline{BC}$  equals the area of the semicircle on the hypotenuse  $\overline{AB}$ ). If Problem 1 is skipped, these same questions are posed on page 2.1 of the tns document.*
- *In Problem 2, students are calculating the sum of the areas of the lunes using the formula  $LUNES = TRIANGLE + semiAC + semiBC - semiAB$ . Students collect data as they change the shape of  $\triangle ABC$ . Be sure that students drag at least two vertices each time they collect a new data point. The calculations in the spreadsheet will support the conjecture that the sum of the areas of the lunes equals the area of the triangle.*

**Activity extensions**

- *Problem 3 is an Extension that proves the conjecture. If the sides of the triangle are  $a$ ,  $b$ , and  $c$ , the areas of the semicircles can be expressed as  $\pi\left(\frac{a}{2}\right)^2$ ,  $\pi\left(\frac{b}{2}\right)^2$ , and  $\pi\left(\frac{c}{2}\right)^2$ . When these expressions are substituted into the formula above and simplified, the conjecture can be proven with the Pythagorean Theorem.*
- *Hippocrates was a Greek mathematician who examined the lune problem. His figure was a square inscribed in a circle, with another semicircle constructed on one side of the square. Create this figure, and show that the area of the lune on one side of the square is equal to the area of the isosceles right triangle that is one-quarter of the square.*

**References**

Nelson, David (Ed.), *Penguin Dictionary of Mathematics, Second Edition*, 1998.

Wells, David, *The Penguin Dictionary of Curious and Interesting Geometry*, 1991.

---

Student TI-Nspire Document *Lunes.tns*

1.1 1.2 1.3 Lunes

**The Lunes of Hippocrates**

Geometry

1.1 1.2 1.3 Lunes

On page 1.3, construct right triangle ABC beginning with a segment and a perpendicular line.

Construct circles with diameters AB, BC, and AC.

Find the area of the triangle and the circles, and calculate the areas of each of the semicircles.

1.1 1.2 1.3 Lunes

1.2 1.3 2.1 Lunes

On page 2.2, right triangle ABC is constructed. Circles are constructed with diameters AB, BC, and AC.

The areas of the triangle and the 3 circles are measured, and the areas of the semicircles are calculated.

Do you notice a relationship between the areas of the 3 semicircles? Drag point A, B or C and observe.

1.3 2.1 2.2 Lunes

tri = 6.99 cm<sup>2</sup>     circle     7.51 cm<sup>2</sup>  
    2     semiAC = 3.76

16.06 cm<sup>2</sup>  
 semiBC = 8.03

23.57 cm<sup>2</sup>  
 semiAB = 11.79

2.1 2.2 2.3 Lunes

The lunes are the regions *inside* the two small semicircles (with diameters AC and BC) but *outside* the large semicircle (with diameter AB).

The sum of their areas is found by adding and subtracting other areas. What areas would you add and subtract to find the sum of the areas of the lunes?

2.2 2.3 2.4 Lunes

On page 2.5, the area values are stored as variables. Page 2.6 is a spreadsheet set to capture area data.

Press **CTRL** and **decimal point** to capture a set of values.

Return to page 2.5 and drag point A, B or C to a new location. Press **CTRL decimal point**. Collect at least 5 data points. You do not need to change pages while collecting.

2.3 2.4 2.5 Lunes

tri = 6.99 cm<sup>2</sup>     circle     7.51 cm<sup>2</sup>  
    2     semiAC = 3.76

16.1 cm<sup>2</sup>  
 semiBC = 8.03

23.6 cm<sup>2</sup>  
 semiAB = 11.8

2.4 2.5 2.6 Lunes

A	tr	ac	bc	ab	lunes
1	=captu	=captu	=captu	=captu	
2					
3					
4					
5					

2.5 2.6 2.7 Lunes

Examine the data in the spreadsheet.

What do you notice about the area of the lunes and another area already found?

Advance to page 2.8. Drag point A, B or C to change the triangle's shape and observe if your conjecture is true.

2.6 2.7 2.8 Lunes

tri = 6.99 cm<sup>2</sup>     circle     7.51 cm<sup>2</sup>  
    2     semiAC = 3.76

16.1 cm<sup>2</sup>  
 semiBC = 8.03

23.6 cm<sup>2</sup>  
 semiAB = 11.8

w+x+y-z  
 lunes = 6.99

2.7 2.8 3.1 Lunes

Now let's look at the algebra of the areas involved in the lunes figure:

Sum of areas of Lunes =  
 tri + semiAC + semiBC - semiAB

Let a = length of side BC, b = length of side AC, and c = length of side AB.

Write expressions for the areas of the triangle and semicircles in terms of a, b, and c.

2.8 3.1 3.2 Lunes

Let a = length of side BC, b = length of side AC, and c = length of side AB.

area tri =  
 area semiAC =  
 area semiBC =  
 area semiAB =

3.1 3.2 3.3 Lunes

Substitute your expressions into the equation below and simplify.

Lunes = tri + semiAC + semiBC - semiAB

Does this prove the conjecture from problem 2?