The Power of Trigonometric Integrals

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Student Activity

7 8 9 10 11 **12**

TI-Nspire™ | Activity | Student | 4 hours

Introduction

In this activity you will:

- evaluate definite integrals involving powers of trigonometric functions,
- derive a recurrence relation,
- Define a function to verify the results.

Definite integrals involving powers of the sine function.

Question: 1.

a) Evaluate each of the following:

i) $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$

ii) $\int_0^{\frac{\pi}{2}} \sin^2(x)\,dx$

iii) $\int_0^{\frac{\pi}{2}} \sin^3(x)\,dx$

iv) $\int_0^{\frac{\pi}{2}} \sin^4(x)\,dx$

v) $\int_0^{\frac{\pi}{2}} \sin^5(x)\,dx$

b) A recurrence relation is an equation that recursively defines a sequence, the results in Part (a) form such a sequence.

i) Let $S(n)=\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx$ use integration by parts to show that $S(n)=\frac{n-1}{n}S(n-2)$.

ii) Use the recurrence relation, established in the previous question, to check your answers to (a)(iii) and (a)(v).

iii) Use the recurrence relation to find, $S(6), S(7), S(8), S(9), S(10)$

iv) Use CAS to check the values of $S(n)$ for $n=1,2,....,10.$

v) Graph the results for $S(n)$ versus *n*, for $n=1,2,....,10.$

vi) Verify that for *n* even, $S(n)=\frac{\pi}{4}\prod_{j=2}^{\frac{n}{2}}\left(\frac{2j-1}{2j}\right)$ and for *n* odd $S(n)=\prod_{j=1}^{\frac{n-1}{2}}\left(\frac{2j}{2j+1}\right)$, and hence write a TI-Nspire function (not involving definite integrals) to evaluate $S(n)$.

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Definite integrals involving powers of the cosine function.

Question: 2.

a) Use graphs to help explain why $\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \int_0^{\frac{\pi}{2}} \sin(x)\,dx$

b) Let $C(n) = \int_0^{\frac{\pi}{2}} \cos^n(x)\,dx$, show that $C(n) = S(n)$ for $n = 1, 2, ..., 5.$

c) Show that $C(n) = S(n)$ for all $n \in Z.$

Definite integrals involving powers of the tangent function.

Question: 3.

a) Evaluate each of the following:

i) $\int_0^{\frac{\pi}{4}} \tan(x)\,dx$

ii) $\int_0^{\frac{\pi}{4}} \tan^2(x)\,dx$

iii) $\int_0^{\frac{\pi}{4}} \tan^3(x)\,dx$

iv) $\int_0^{\frac{\pi}{4}} \tan^4(x)\,dx$

v) $\int_0^{\frac{\pi}{4}} \tan^5(x)\,dx$

b) Let $T(n) = \int_0^{\frac{\pi}{4}} \tan^n(x)\,dx$ show that $T(n) = \frac{1}{n-1} - T(n-2)$. [Do not use integration by parts]

c) Use the recurrence relation obtained in the previous question to check your answers to Q3(a).

d) Use the recurrence relation to find: $T(6), T(7), T(8), T(9), T(10), T(11), T(12), T(13).$

e) Use CAS to check the values of $T(n)$ for $n = 1, 2, ..., 13.$

f) Graph the results for $T(n)$ versus *n*, for $n = 1, 2, ..., 10.$

g) Define the function shown here and use it to verify that

when *n* is divisible by 4: $T(n) = \sum_{k=0}^{\frac{n}{2}-1} \left(\frac{(-1)^{k+1}}{2k+1} \right) + (-1)^{\frac{n}{2}} \frac{\pi}{4}$,

when *n* even and not divisible by 4: $T(n) = \sum_{k=0}^{\frac{n}{2}-1} \left(\frac{(-1)^{k}}{2k+1} \right) + (-1)^{\frac{n}{2}} \frac{\pi}{4}$,

and, when *n* is odd: $T(n) = (-1)^{\frac{n-1}{2}} \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{(-1)^{k}}{2k} \right) + (-1)^{\frac{n-1}{2}} \frac{1}{2} \log_e(2).$

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