A Special Function

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Student Activity - Answers

7 8 9 10 **11 12**

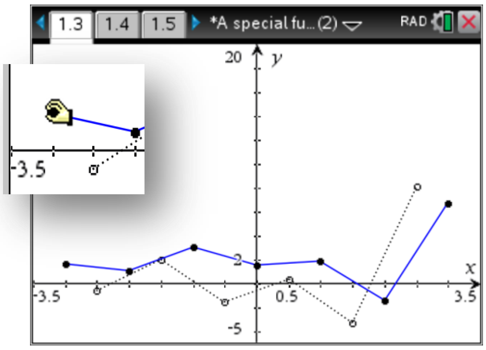
TI-Nspire | Investigation | Student | 30 min

Introduction

Derivatives of a polynomial function are relatively straight forward, but what about logarithms, exponentials, trigonometric functions and other special functions that are commonly used in mathematics? This activity explores the derivative of a unique function with the property that the derivative is exactly the same as the original function.

A Flexible Function

Open the TI-Nspire document “A Special Function”.
Read the instructions on page 1.2 and then navigate to page: 1.3.
To move a point on the solid line, place the mouse over the point and then ‘click and hold’ or press Ctrl + Click to grab the point. To release the grip, press [ESC].
Move the point with a view to making the dotted line as close as possible to the solid line in that region. Once the first point has been positioned, move on to the next.



The dotted or gradient line is aligned to the midpoint of each moveable segment. The only way to move the dotted gradient line is by moving segments on the solid line.

Question: 1.

Describe the general shape of your solution.

Students may draw a graph to illustrate the general shape or describe it as ‘exponential’ in shape.

Question: 2.

Is your solution unique?

The solution is not unique. Students should realise they can translate the graph parallel to the x axis and achieve a similar outcome or reflect in the x axis, or produce the simple case of y = 0. All the exponential solutions are of the form: ke^x or $\pm e^{x+c}$.

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A Polynomial Model

Consider a polynomial defined as: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, the special function has the property: $p(x) = p'(x)$. Given that $p'(x) = p''(x)$ then it follows $p(x) = p''(x)$ and similarly: $p(x) = p'''(x)$ and so on.

Question: 3.

Determine an expression for: $p(0)$

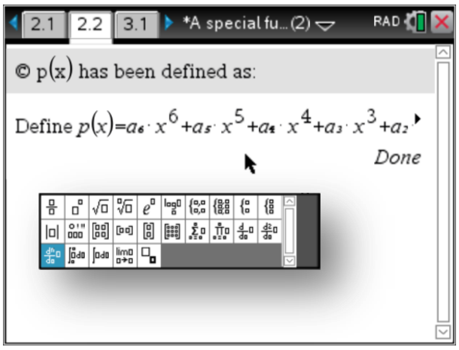
$p(0) = a_0$ … Substitution of x = 0 reduces the expression to the constant term.

Question: 4.

Determine an expression for: $p'(0)$

$p'(0) = a_1$ … The derivative removes the first constant, substitution of x = 0 leaves only a_1

Read the information on page 2.1 then navigate to page 2.2. Use the derivative template and compute the derivative of $p(x)$ as defined on the calculator.



The template provides several derivative templates; the n[th] derivative is a very efficient tool to use in this problem.

For the following questions assume that: $p(0) = 1$.

In each question, store the computed value in the appropriate variable.

Question: 5.

Given $p(0) = 1$ determine the value of a_0 .

$p(0) = a_0 \Rightarrow a_0 = 1$

Question: 6.

If $p(0) = 1$ then it follows: $p'(0) = 1$, hence determine the value of a_1 .

$p'(0) = a_1 \Rightarrow a_1 = 1$

Question: 7.

Given that: $p''(0) = 1$, hence determine the value of a_2 .

$p''(0) = 2a_2 \Rightarrow a_2 = \frac{1}{2}$

Question: 8.

Repeat the steps above to determine values for: a_3, a_4, a_5, a_6 .

$a_3 = \frac{1}{6},\ a_4 = \frac{1}{24},\ a_5 = \frac{1}{120},\ a_6 = \frac{1}{720}$

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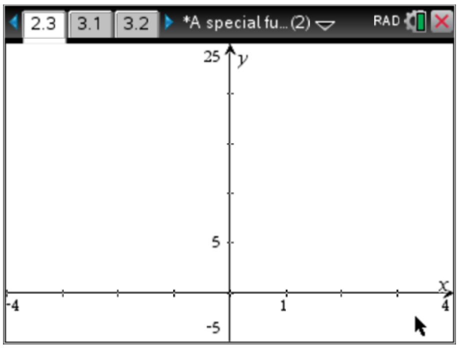
Navigate to page 2.3 (Graph Application).

If all the coefficients have been stored (defined) the graph of $p(x)$ will appear.



Question: 9.

Comment on the general shape of the function, particularly where $x < 3$, and the degree of the polynomial.

General shape is similar to the one first obtained, an exponential; however when $x < 3$ the graph turns back upward, as expected for a polynomial with an even degree.

Question: 10.

Determine a rule $a(n)$ for the general coefficient: a_n

[Do not store the general expression for a_n]

$a_n = \dfrac{1}{n!}$ This can be seen by the formation of the coefficient during repeated differentiation.

Question: 11.

Use the general rule for the coefficient and the summation tool (calculus menu) to define the rule for $p(x)$ as a polynomial degree 12 and use it to evaluate $p(1)$ correct to 10 decimal places.

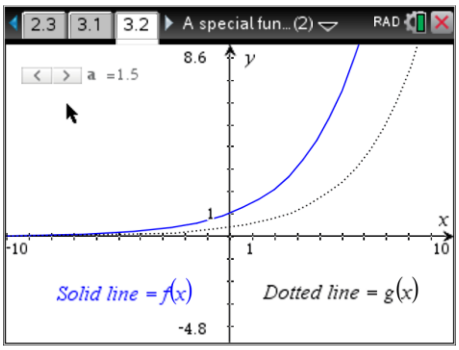
ie: $p(x) = \sum_{n=0}^{12}\left(a(n) \cdot x^n\right)$

$p(x) = \sum_{n=0}^{12}\left(\dfrac{x^n}{n!}\right) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!} ...$ $p(1) \approx 2.7182818282$

An Exponential Function

In this section an exponential function is used to replace the original discrete, flexible line. The exponential function is of the form $f(x) = a^x$ and can be used to explore the original problem, identify a function such that: $f(x) = f'(x)$.

Navigate to page 3.2 and adjust the slider until the gradient line (dotted) sits as close as possible to the solid line (original function).



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Question: 12.

Determine the approximate value of *a* that produces an approximation for: $f(x) = g(x)$.

Approximate value: 2.7 based on slider value.

Question: 13.

Use the value determined for $p(1)$ in question 11 as the base. The base value (slider) can be changed by clicking on the value and editing. Once this value has been changed, press Ctrl+ T to produce a table of values and comment on how close the original function and its derivative appear.

Table values are the 'same' within the accuracy displayed.

Question: 14.

The value computed for $p(1)$ is an approximation for a transcendental number; identify the number and state special function such that the derivative function is the same as the original. (Test your answer using the calculator)

Euler's number: e Calculator returns: $\frac{d(e^x)}{dx} = e^x$ and also $\frac{d(ke^x)}{dx} = ke^x$

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