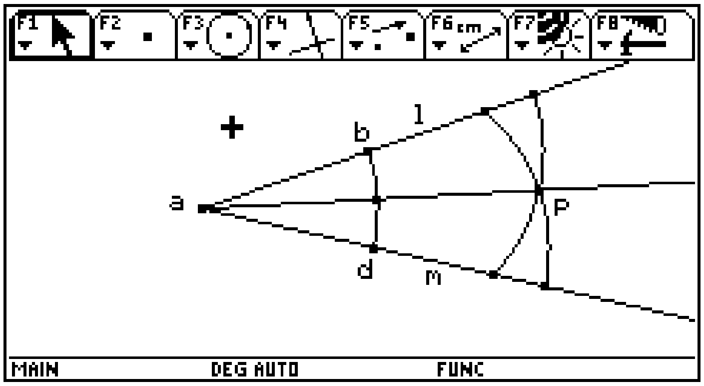
How to Find the Equation of an Angle Bisector Using a Construction



a, b, d, and p represent points A, B, D, and P. 1 and m represent the slopes of the lines.

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I like to assign messy algebra problems to my advanced classes because I believe they help students improve their algebraic accuracy and give them a sense of confidence when they complete a page-long problem correctly.

One problem I give asks the students to find the equation of a bisector of the angle formed by two intersecting lines. My students this year have not had a formal introduction to trigonometry and therefore I wasn't anticipating that any would use trig to solve the problem.

Students typically write the equations of two general lines L [y - b = m (x - a)] and M [y - b = n(x - a)] which have two different slopes and intersect at the point A(a,b). They then consider a point P with coordinates (u, v), find the distances from P to L and from P to M and then set these distances equal to each other to find the coordinates of P in terms of a, b, n, and m. They use the property that the angle bisector is equidistant from the sides of the angle in order to state that the line AP is an angle bisector of one of the angles formed by L and M. This method takes about a page to write.

This year I assigned the problem at the same time I was teaching geometric constructions. The traditional method for constructing an angle bisector is to construct a circle of arbitrary radius whose center is the vertex of the angle. Then, using the two points where that circle intersects the rays of the angle as centers, construct two circles with the same initial radius. The point where these two new circles intersect and the vertex of the original angle determines the angle bisector.

We can use this construction strategy to find the algebraic equation of the angle bisector in the following way. Our two lines L [y - b = m (x - a)] and M [y - b = n(x - a)] have two different slopes and intersect at the point A(a,b). Consider circle C with center A(a,b) and radius 1. The equation of circle C is $(x - a)^2 + (y - b)^2 = 1$. This circle intersects line L at the point B $\left(a+\sqrt{\frac{1}{1+m^2}},\ b+m\sqrt{\frac{1}{1+m^2}}\right)$

and the line M at the point D $\left(a+\sqrt{\frac{1}{1+n^2}},\ b+n\sqrt{\frac{1}{1+n^2}}\right)$

Now, if we were to construct circles with radius 1 centered at B and D, they would intersect at point A and at a fourth point, E, on the angle bisector. This fourth point is also the fourth vertex of a parallelogram whose other three vertices are A, B, and D. Therefore, we can find the coordinates of E by adding the vectors AB and AD together to get E

$$\left(a+\sqrt{\frac{1}{1+n^2}}+\sqrt{\frac{1}{1+m^2}},\ b+n\sqrt{\frac{1}{1+n^2}}+m\sqrt{\frac{1}{1+m^2}}\right)$$

The slope of AE, the angle bisector, is therefore:

$$\frac{n\sqrt{\frac{1}{1+n^2}}+m\sqrt{\frac{1}{1+m^2}}}{\sqrt{\frac{1}{1+n^2}}+\sqrt{\frac{1}{1+m^2}}}$$

and the equation of AE is

$$y-b=\left(\frac{n\sqrt{\frac{1}{1+n^2}}+m\sqrt{\frac{1}{1+m^2}}}{\sqrt{\frac{1}{1+n^2}}+\sqrt{\frac{1}{1+m^2}}}\right)(x-a)$$

We can verify that the slope is correct using trigonometry. If θ is the angle line L makes with the x-axis then $\tan\theta = m$. If α is the angle line M makes with the x-axis then $\tan\alpha = n$. $\tan\theta = m$ implies that

$$\sin\Theta=\frac{m}{\sqrt{1+m^2}} \quad \text{and} \quad \cos\Theta=\frac{1}{\sqrt{1+m^2}}$$

with similar results for α in place of θ. Assuming that $\theta > \alpha$ then the angle between the lines is $\theta - \alpha$ and then we can apply the trig identity:

$$\tan\left(\frac{\theta-\alpha}{2}\right)=\frac{\sin\theta+\sin\alpha}{\cos\theta+\cos\alpha}=\frac{\frac{m}{\sqrt{1+m^2}}+\frac{n}{\sqrt{1+n^2}}}{\frac{1}{\sqrt{1+m^2}}+\frac{1}{\sqrt{1+n^2}}}$$

which is what we arrived at above.