EXPLORATIONS

Chapter 5

Harmonic Motion

In the last chapter, you modeled the motion of an object when the only force acting on the object was gravity. In this chapter, you will consider the case where there is another major force acting on the object.

Introduction

If a mass m is attached to the end of a suspended spring, both gravity and the spring exert a force on the mass. If the mass is left undisturbed, the force of the spring will balance the force of gravity and the mass will hang motionless. This is called the *equilibrium point* of the spring. If you pull the spring down from the equilibrium point, the net force on the mass will be in the upward direction. If you push the mass up from the equilibrium point, the net force on the mass will be in the downward direction. As the mass moves further from the equilibrium point, the net force on the mass will increase.

Hooke's law describes the net force on the mass. This law is $F = -ks$, where F is the net force exerted on the mass, s is the displacement of the spring and k is a constant that depends on the stiffness of the spring. The negative sign in Hooke's law indicates the force of the spring is in the opposite direction of the displacement.

If you pull the mass down from the equilibrium point and let go, the mass will oscillate up and down. This is called *harmonic motion*. If the mass m of the hanging object is very large compared to the mass of the spring, you can combine Hooke's law with Newton's second law of motion ($F = ma$) to obtain a differential equation that describes the displacement of the mass. This equation is:

$$s'' = -\frac{k}{m}s.$$

What should the graph of displacement look like? What is the equation that describes displacement as a function of time? In the next example, you will use the differential equation to generate data that model the displacement of the mass and then curve fit the data to find the equation for displacement.

Example 1: Fitting an Equation to a Graph of Spring Motion

If

$$\frac{k}{m} = 4$$

the differential equation is

$$s'' = -4s$$

for the displacement s of a mass attached to a hanging spring. If $s(0) = -3$ (initial displacement = 3 down) and $s'(0) = 0$ (initial velocity = 0), graph the displacement of the spring over time and then find the equation for displacement as a function of time.

Solution

Make the following substitutions to reduce the differential equation to a system of first order equations.

displacement	$= s = Q1$
velocity	$= s' = Q'1 = Q2$
acceleration	$= s'' = Q''1 = Q'2 = -4Q1$

1. Enter the variables shown in Figures 5.1 through 5.6 to graph the solution to the differential equation.

Figure 5.1

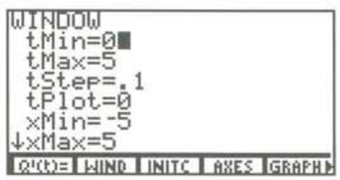

Figure 5.2

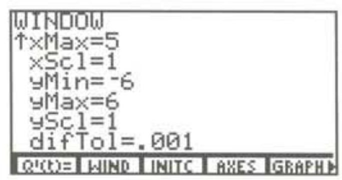

Figure 5.3

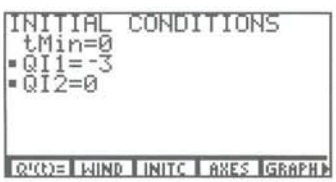

Figure 5.4

Figure 5.5

Figure 5.6

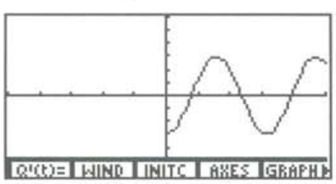

Figure 5.7

Did you expect the graph in Figure 5.7? You are going to use the **SinR** (Sinusoidal Regression) feature of the TI-86 to find an equation for this graph. In order to use regression, you need lists containing the x- and y- coordinates of points on the graph. You can generate these with the **DrEqu** feature found in the GRAPH DRAW menu. This feature is similar to the **EXPLR** (Explore) feature in the GRAPH menu in that it allows you to specify initial conditions graphically and then graph the solution corresponding to these initial conditions. **DrEqu** is different from **EXPLR** in that you can save the points generated on the graph in two lists.

1. Press [GRAPH] [MORE] [F2] (**DRAW**) to display the **DrEqu** option. (Figure 5.8)

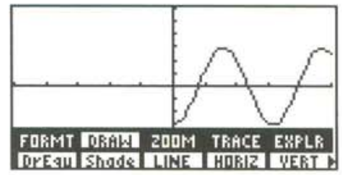

Figure 5.8

2. Press [F1] to paste this command to the home screen. (Figure 5.9)

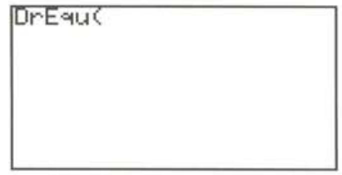

Figure 5.9

3. **DrEqu** uses four parameters. The first two are the axes used and the last two are the names of the lists to store the x- and y- coordinates of the graph points. The axes are **t** and **Q1,** and the two lists are **L1** and **L2.** To enter these parameters, press [2nd] [ALPHA] [T] [,] [ALPHA] [Q] [1] [,] [ALPHA] [L] [1] [,] [ALPHA] [L] [2] [)]. (Figure 5.10)

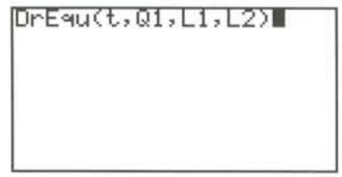

Figure 5.10

4. When you press [ENTER], you will see the cursor blinking at the center of the screen, and the cursor coordinates appear at the bottom of the screen. (Figure 5.11)

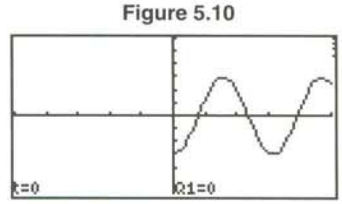

Figure 5.11

5. Move the cursor as close as possible to the initial coordinates **t = 0, Q1 = $^-$3**. (Figure 5.12)

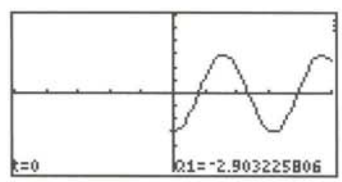

Figure 5.12

6. When you press [ENTER], a graph with this initial point will be drawn in thick style. (Figure 5.13)

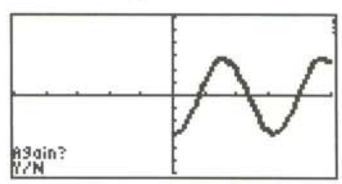

Figure 5.13

7. Since you don't want to draw another graph, press [N]. The thick graph is essentially the same one you already had. The reason you used **DrEqu** to redraw the graph is so you could store the x and y coordinates of points on this graph in lists **L1** and **L2**. You can then use these lists and the sinusoidal regression feature of the TI-86 to find a sine function that fits the graph.

8. Press [2nd] [STAT] [F1] (**CALC**) [MORE] [F2] (**SinR**) to paste the sinusoidal regression command to the home screen. (Figure 5.14)

Figure 5.14

9. **SinR** has three parameters. The first two parameters are the lists containing the data to be analyzed. The third parameter is the name of the variable used to store the regression equation. (This third parameter is optional.) Enter these parameters by pressing [ALPHA] [L] [1]Π 1[L] Z Π – 1 [Y] Ψ (Figure 5.15)

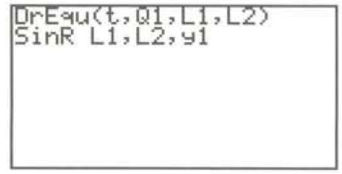

Figure 5.15

10. Press [ENTER] and, after a pause, the coefficients of the sine function that best fits the data are displayed on the screen. (Figure 5.16)

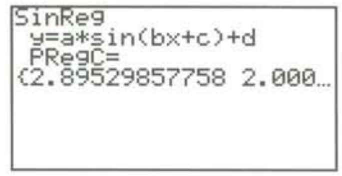

Figure 5.16

11. You can see the other coefficients by scrolling to the right using [▸].

    The sine function that models the oscillating mass appears to be approximately

    $$y = 2.895\sin(2x - 1.571)\,,$$

    where y is displacement and x is time. This function has been stored in $y1$.

Figure 5.17

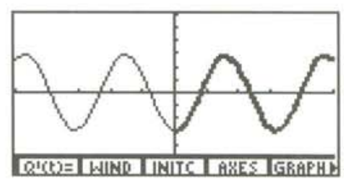

Figure 5.18

12. If you select **Radian** on the mode settings screen and then enter the command **DrawF y1** on the home screen, the graph of **y1** will be drawn on the same screen with the graph of **Q1**. Since the two graphs appear to coincide for values of x greater than zero, you have found the equation for displacement of the mass as a function of time. (Figures 5.17 and 5.18)

In the next example, you will examine the relationship between displacement and velocity for the oscillating mass.

Example 2: Phase Trajectories for Spring Motion

If you use the same setup and initial conditions as Example 1, what will be the effect on the graph of changing the axes to **x** = Q1, **y** = Q2? (Figure 5.19)

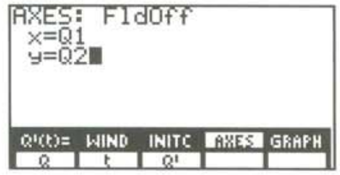

Figure 5.19

Solution

Recall that **Q1** = displacement and **Q2** = velocity. The initial displacement is $^-3$ and the initial velocity is 0, so the graph should start at the point ($^-3$,0). The mass begins to move up while the velocity increases until the mass reaches the equilibrium point. After passing the equilibrium point, the mass continues to move up but the velocity decreases because the spring opposes the motion of the mass. This means x should increase while y first increases and then decreases.

What will happen after the mass reaches the top of its path? Sketch the graph of displacement versus velocity. Support your sketch with a graph by making the changes shown in Figure 5.18 and graphing the solution.

Does your graph look like Figure 5.20?

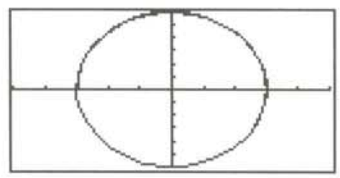

Figure 5.20

Review the solution to Example 2 until you understand why the graph is a circle. The plane in which this graph is drawn is called a *phase plane* and the graph is called a *phase trajectory* or an *orbit*.

1. You can see the effect of various initial conditions with the **DirFld** (Direction Field) feature. Select this mode by highlighting **DirFld** i*on the format screen and pressing [ENTER]. (Figure 5.21)

Figure 5.21

2. Select the axes editor and verify that **x** = Q1 and **y** = Q2. Notice this menu shows that the **Direction Field** has been selected. The **fldRes** variable determines the resolution of the direction field. The **dTime** variable will be discussed in a later chapter. (Figure 5.22)

Figure 5.22

3. Press [2nd] [F5] (**GRAPH**) to see the direction field. (Figure 5.23)

   The direction field is like a slope field. If you start at any point in the phase plane and move so your path is tangent to the nearby line segments, you will trace out an orbit.

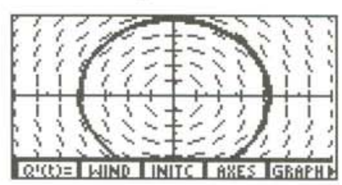

Figure 5.23

4. You can use the **EXPLR** feature in the GRAPH menu to select different initial conditions graphically and see the resulting orbit. When you are finished with the **EXPLR** feature, you can press EXIT to return to the GRAPH menu. (Figures 5.24 and 5.25)

   The smaller circle corresponds to the mass oscillating with a smaller amplitude.

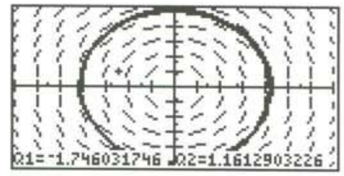

Figure 5.24

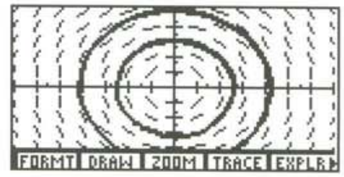

Figure 5.25

In the next example, you will model an effect that dampens the amplitude of the mass as it oscillates.

Example 3: Damped Motion

In the previous spring example, you ignored forces of friction that might oppose the motion of the oscillating mass. Air resistance provides such a frictional force. If you assume the force due to air resistance is directly proportional and opposed to the velocity of the mass you obtain the following differential equation for displacement s:

$$s'' = -\frac{k}{m}s - \frac{c}{m}s'.$$

In this equation k is the spring constant, m is the mass of the object and c is the constant of proportionality for air resistance. If

$$\frac{k}{m} = 4 \text{ and } \frac{c}{m} = 0.5$$

predict the motion of the spring. Support your prediction graphically.

Solution

The system of differential equations that includes air resistance is

displacement	$= s = Q1$
velocity	$= s' = Q'1 = Q2$
acceleration	$= s'' = Q''1 = Q'2 = -4Q1 - 5Q2$

1. Enter this system in the differential equation editor (**Q(t) =** screen). (Figure 5.26)

Figure 5.26

2. Select **FldOff**. (Figure 5.27)

Figure 5.27

3. Select **x** = t, **y** = Q1 in the axes editor. (Figure 5.28)

Figure 5.28

4. Keep the same viewing window and initial conditions as Example 1 and then graph the solution of displacement versus time. (Figure 5.29)

   Did you predict the damped oscillations?

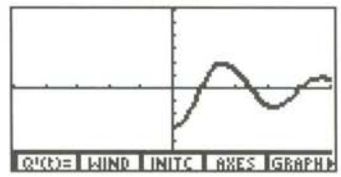

Figure 5.29

   What will the phase plane and orbit look like for the damped oscillations? Sketch your prediction.

5. Turn the direction field (**DirFld**) on in the format screen and select **x** = Q1, **y** = Q2 in the axes editor. (Figures 5.30 and 5.31)

Figure 5.30

Figure 5.31

Does your sketch look like Figure 5.32?

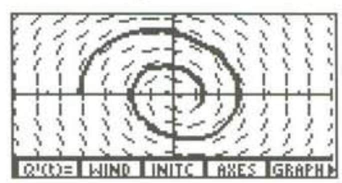

Figure 5.32

Why does the orbit spiral in? What will happen if you increase **tMax** in the window editor? (Figure 5.33)

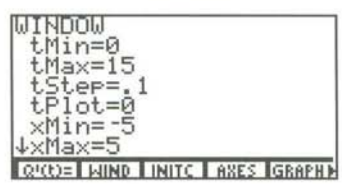

Figure 5.33

The spiral into the origin shows the spring reaching equilibrium as the amplitude of the oscillations approaches zero. (Figure 5.34)

Figure 5.34

You can see this effect in the displacement versus time graph by turning **FldOff** and changing **xMax** to 15. It appears the mass reaches equilibrium after about four oscillations. (Figure 5.35)

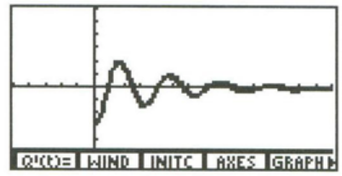

Figure 5.35

Remember this graph shows how displacement is related to time but it does not show the actual motion of the mass.

In the last example, you will learn how to model the actual motion of the mass.

Example 4: Animation

Create a model for the actual motion of the mass and spring (including air resistance) that was described in Example 3.

Solution

This graph should show the mass oscillating straight up and straight down. This is essentially an animation. You can create this effect with the animation graphing style by making the selections in the differential equation editor (**Q'(t)** = screen) and the axes editor shown at the right.

Figure 5.36

1. Enter **Q'3** = 0 in the differential equation editor and select the animation graphing style. (Figure 5.36)
2. Enter **QI3** = 1 in the initial conditions editor. (Figure 5.37)

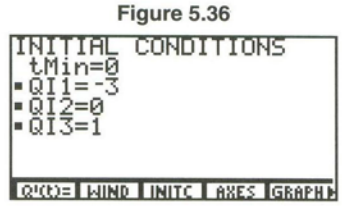

Figure 5.37

3. Select **Euler** and **FldOff** on the format screen. (Figure 5.38) Selecting **Euler** sacrifices some accuracy in the solution to the differential equation but makes the animation run more smoothly.

   Selecting **Euler** mode creates a new variable in the window editor called **EStep**. The current value of **EStep** is 1. Larger values for **EStep** will result in better accuracy but slower graphs. (Figure 5.39)

Figure 5.38

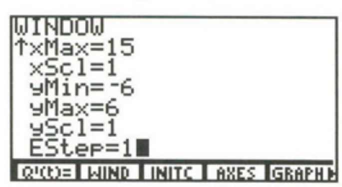

Figure 5.39

4. Select **x** = Q3 and **y** = Q1 in the axes editor. (Figure 5.40)

Figure 5.40

When you select **x** = Q3 and **y** = Q1 in the axes editor, you will produce a graph in which the x-coordinate is always 1 and the y-coordinate is the displacement. Since the graphing style for **Q3** is animation, you should see an animation of the mass when you press [2nd] [GRAPH]. The animation shows the damped oscillations of the mass. (Figures 5.41 and 5.42)

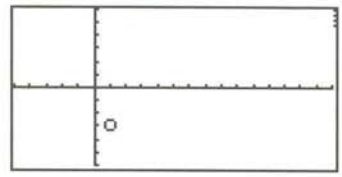
Figure 5.41

When the animation is finished, the circle disappears. If you would like to see the animation again, select **ClDrw** in the GRAPH DRAW menu. This clears the graph screen and then redraws the graph.

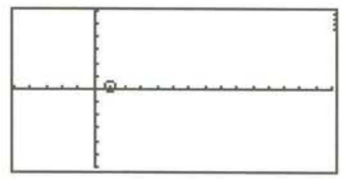
Figure 5.42

Now use what you have learned in this chapter to model the following harmonic motion problems.

Exercises

In the following exercises, be sure to use **RK** mode in the GRAPH FORMAT menu for any graphs other than animations. Since oscillations are very sensitive to the numerical method used, you will need to use **RK** mode or **Euler** mode with a very large value for **EStep** for the animations, otherwise the oscillations may diverge instead of converging.

1. An oscillating spring is modeled by the differential equation

   $s'' = -6s \quad s(0) = 4 \quad s'(0) = 0$, where *s* is displacement.

   a. In differential equation graphing mode, graph displacement versus time.

   b. Regraph the solution using Draw Equation (**DrEqu**) and store the **x** and **y** coordinates in lists **L1** and **L2**.

   c. Use Sinusoidal Regression (**SinR**) on the lists to find the analytic solution to the differential equation.

   d. Use Draw Function (**DrawF**) to compare the solution in **c** with the solution in **a**.

   e. Animate the actual motion of the spring.

   f. Draw the phase plane and orbit for displacement versus velocity.

2. An oscillating spring is modeled by the differential equation

   $s'' = -5s \quad s(0) = -2 \quad s'(0) = 0$, where *s* is displacement.

   a. In differential equation graphing mode, graph displacement versus time.

   b. Regraph the solution using Draw Equation (**DrEqu**) and store the **x** and **y** coordinates in lists $L1$ and $L2$.

   c. Use Sinusoidal Regression (**SinR**) on the lists to find the analytic solution to the differential equation.

   d. Use Draw Function (**DrawF**) to compare the solution in **c** with the solution in **a**.

   e. Animate the actual motion of the spring.

   f. Draw the phase plane and orbit for displacement versus velocity.

3. A damped oscillating spring is modeled by the differential equation

   $s'' = -6s - .2s' \quad s(0) = 4 \quad s'(0) = 0$, where *s* is displacement.

   a. In differential equation graphing mode, graph displacement versus time.

   b. How long will it take for the oscillations to diminish to one-fourth of their original amplitude?

   c. Draw the phase plane and orbit for displacement versus velocity.

4. An oscillating spring is modeled by the differential equation

$$s'' = -6s - 5s' \quad s(0) = 4 \quad s'(0) = 0$$, where s is displacement.

a. In differential equation graphing mode, graph displacement versus time.

b. Draw the phase plane and orbit for displacement versus velocity.

c. This spring is said to be overdamped. Describe the motion of an overdamped spring. Animate the motion of the spring to confirm your description. What could cause this type of damping?