Investigating Properties of Trapezoids

Definitions:

Trapezoid — a quadrilateral with at least one pair of parallel sides (Figure 1).

Isosceles trapezoid — a trapezoid with one pair of base angles and the non-parallel pair of sides equal in measure.

Tessellation – a covering or **tiling** of a plane with a pattern of figures so there are no overlaps or gaps.

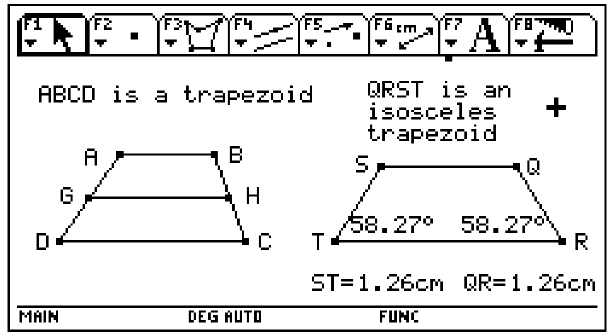

Figure 1

Construct and Investigate:

1. Determine three ways to construct a trapezoid using the TI-92. Test your constructions by dragging independent points of the trapezoids to be sure that the figures remain trapezoids at all times.
2. Find a relationship regarding angle measurement that appears to be true for all trapezoids. Write a conjecture stating this relationship. Test your conjecture by dragging a vertex of the trapezoid and observing the relationship for a variety of different shaped trapezoids.
3. Determine three ways to construct an isosceles trapezoid on the TI-92. Write a step-by-step procedure for each method and test these procedures before moving to the next activity.
4. State at least three properties that are true for all isosceles trapezoids. These properties could include statements about angle measures, side lengths, diagonals, area, or perimeter.

Explore:

1. The area of a trapezoid is given by the formula $A = \frac{b_1 + b_2}{2} h$, where b_1 and b_2 are the lengths of the two bases and h is the height of the trapezoid. Drag one of the non-parallel sides of a trapezoid toward the other. Explain how you can derive the area formula for a triangle from the area formula for a trapezoid.
2. Triangles and trapezoids are related in other ways. The **midsegment of a triangle** is a segment connecting the midpoint of any two consecutive sides of a triangle. The **midsegment of a trapezoid** is the segment connecting the midpoints of the two non-parallel sides. Find a relationship between the midsegments of triangles and trapezoids. Investigate the relationship between a midsegment of a triangle or trapezoid and the area. Explain algebraically why this is always true for any triangle or trapezoid.
3. The large trapezoid shown in Figure 2 can be dissected into similar copies of itself. Starting with a dissected trapezoid, construct the larger trapezoid shown using transformational geometry tools. What properties of this trapezoid allow this transformation? Explain. Can you find other trapezoids that have the same property? Will these figures tessellate the plane? Explain.

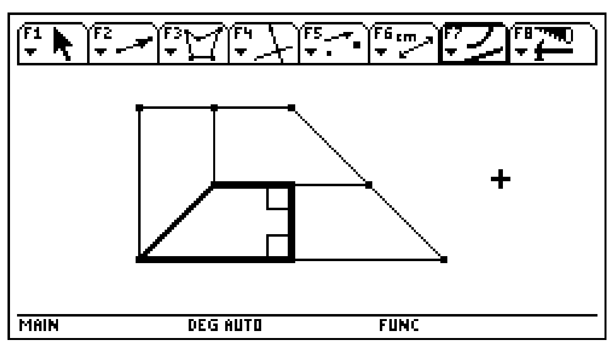

Figure 2

Construct and Investigate:

1. There are many ways to construct a trapezoid. Three ways are listed here, but students may find others that use the properties of trapezoids.
   - Draw two segments connected at a common endpoint. Construct a line parallel to one segment through the non-common endpoint of the other segment. The fourth vertex of the trapezoid can be any point on the parallel line that is in the interior of the angle formed by the first three points (the vertex being the common endpoint).
   - Construct a triangle and draw a line parallel to one side passing through any point on one of the other sides. A trapezoid will be formed by the base, the parallel, and the two non-parallel sides. Draw a polygon over this figure and hide the triangle and the parallel line.
   - Construct a segment, and draw a line parallel to it through a point not on the segment. Connect the endpoints of the segment to any two points on the parallel line so that the non-parallel sides do not cross.
2. As with all convex quadrilaterals, the sum of the interior angles is 360°. The consecutive angles between the parallel sides (bases) of the trapezoid are always supplementary (Figure 3).

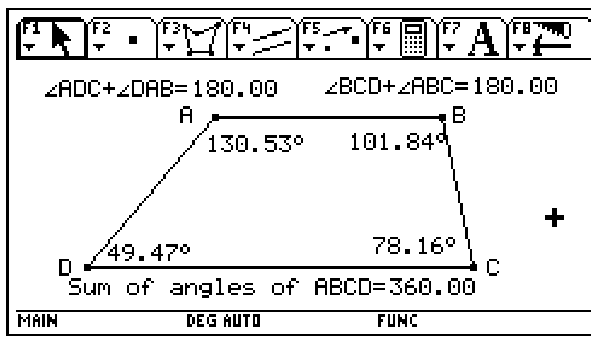

Figure 3

3. Students will find many ways to construct an isosceles trapezoid. Three ways are listed below:
   - Draw a segment and construct its perpendicular bisector. Pick any point on the bisector other than the midpoint of the segment. Construct segments from the endpoints of the original segment to this point forming an isosceles triangle (the base being the original segment). Place another point on the perpendicular bisector between the vertex of the isosceles triangle and the midpoint of the segment. Construct a line through this point parallel to the base of the isosceles triangle. Use the **Polygon** tool to form the isosceles trapezoid using the endpoints of the original segment and the two points of intersection between the parallel line and the sides of the isosceles triangle.
   - Place two points on the screen and construct any line that does not contain either point. Reflect the two points over the line. Use the **Polygon** tool to connect the four points to form an isosceles trapezoid.
   - Construct a line segment. Construct the perpendicular bisector of the segment. Select a point on the perpendicular bisector that is not on the segment. Construct a line parallel to the original segment that goes through this point. Draw a circle with its center at the intersection of the perpendicular bisector and the parallel line. The polygon formed by the endpoints of the original segment and the intersection points of the circle and the parallel line is an isosceles trapezoid.

4. The following properties are true for all isosceles trapezoids:
   - Both pairs of base angles are equal in measure.
   - The perpendicular bisector of one of the bases is also the perpendicular bisector of the other base.
   - The perpendicular bisector of the bases is a symmetry line of the isosceles trapezoid.
   - The diagonals of an isosceles trapezoid are equal in length and divide the trapezoid as follows:
     - Three pairs of congruent triangles (Figure 4).
     - One pair of similar triangles (Figure 5).
   - Isosceles trapezoids are cyclic quadrilaterals, which means the four vertices lie on a circle. Therefore, the product of the diagonals is equal to the sum of the products of the two opposite sides (Ptolemy's Theorem).

     The center of the circumscribed circle is the intersection of the perpendicular bisectors of the two non-parallel sides. This center also lies on the line of symmetry of the trapezoid (Figure 6).

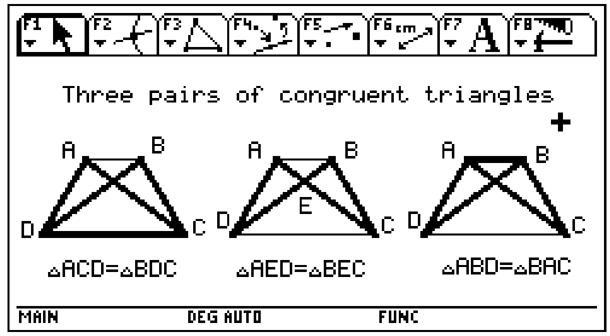

Figure 4

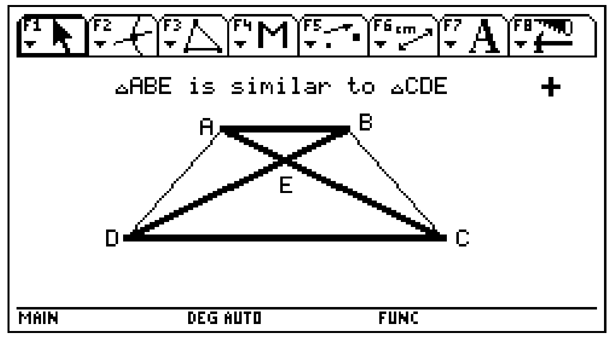

Figure 5

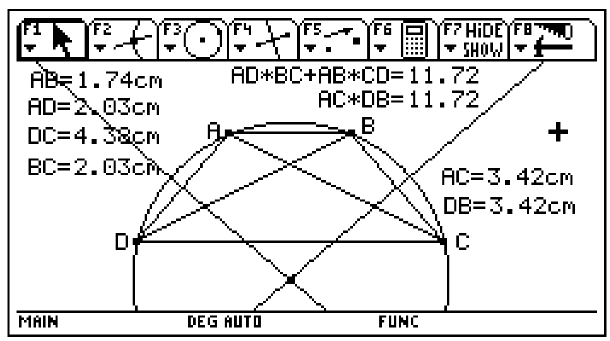

Figure 6

Explore:

1. As the two non-parallel sides of a trapezoid come closer together, the length of the shorter base eventually becomes zero and the trapezoid becomes a triangle (Figures 7 and 8).

   If b_1 represents the shorter base and b_2 represents the longer base, when $b_1 = 0$, the area formula for a trapezoid becomes the area formula for a triangle:

   $$A = ½\,(0 + b_2)\,h = ½\,bh$$

   In a sense, a triangle is a special case of a trapezoid where the shorter base has a length of zero.

2. The length of the midsegment of a trapezoid is equal to the average length of the two parallel sides (Figure 9). The midsegment of a triangle is equal to half the length of its parallel base. If a triangle is considered a special case of a trapezoid with one base zero in length, then the same relationship holds for both figures.

   The area of a trapezoid is the product of the length of its midsegment and its height (Figure 10). The same is true for a triangle.

   If b_3 represents the length of the midsegment of a trapezoid, and you know that $b_3 = \frac{b_1 + b_2}{2}$; and if the area of a trapezoid is $A = ½\,(b_1 + b_2)\,h$, then by substitution, you know that the area of a trapezoid is given by $A = b_3h$.

   In a triangle, the midsegment $b_3 = \frac{b_2}{2}$.

   If the area is given by $A = ½\,bh$, then you know that $A = b_3h$.

   In both cases, the area of the figure is given by the product of the length of its midsegment and its height.

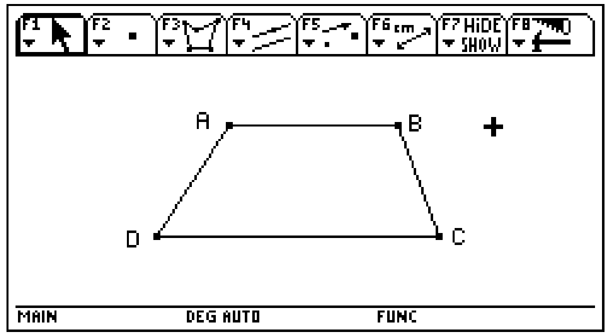

Figure 7

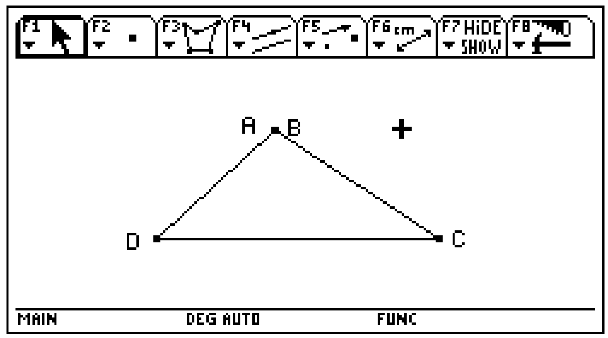

Figure 8

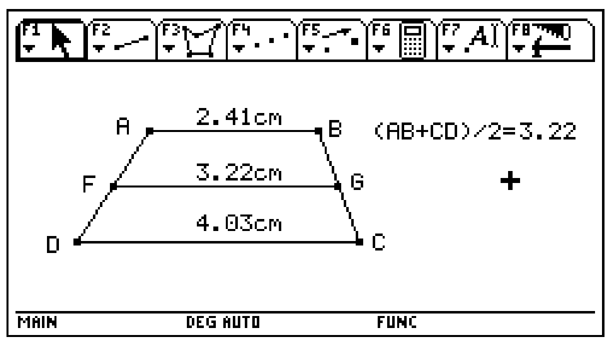

Figure 9

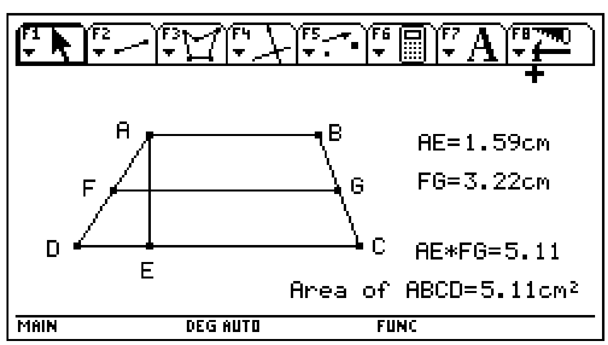

Figure 10

3. Each of the four sub-parts of the whole trapezoid are congruent and similar to the entire figure with a side ratio of 2 and an area ratio of 4.

   Each trapezoid is also special in the following ways:

   - The two bases make right angles to one non-parallel side.
   - The other non-parallel side makes an angle of 45° to one base and 135° to the other base.
   - The longer base is twice as long as the shorter base (Figure 11).

   The construction can be made using reflections and rotations or using reflections and translations.

   Other trapezoids of this type are shown in the figures on the right.

   - Figure 12 is a trapezoid formed by a square and half of a congruent square.
   - Figures 13 and 15 are trapezoids formed by three congruent equilateral triangles so that the angles of the trapezoid are either 60° or 120°.
   - Figure 14 is made up of trapezoids formed by two congruent squares and a 30°-60°-90° triangle.

   These figures are fun to do and can involve many creative and imaginative constructions. Once they determine the original trapezoid, see if your students can construct each one using only isometries.

   Because each example can be dissected or expanded into an infinite number of identical parts, all of them can tessellate the plane.

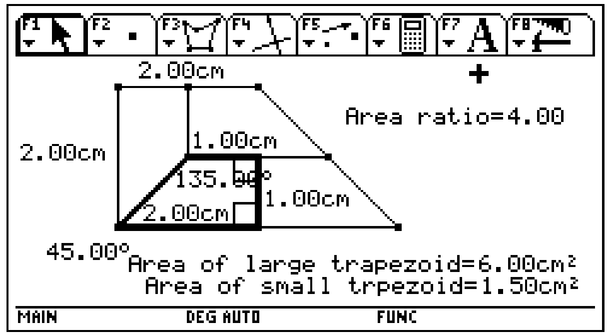

Figure 11

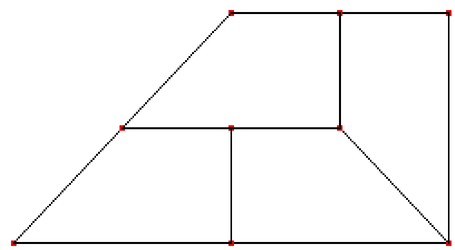

Figure 12

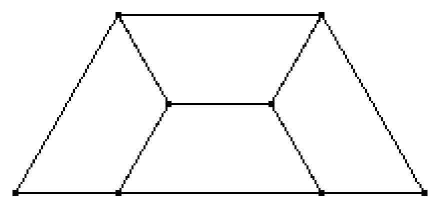

Figure 13

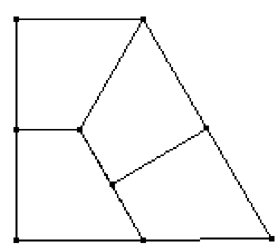

Figure 14

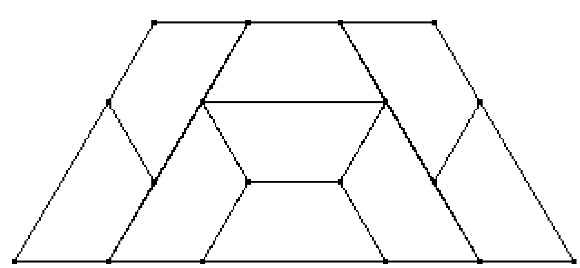

Figure 15