Transformers – ID: 8776

Time required
60 minutes

Topic: Matrices

- *Use matrices to apply isometric transformations to polygons in the plane.*
- *Use matrices to apply successive transformations in the plane.*

Activity Overview

In this activity, students explore a special subset of the transformations of a square called the symmetry group. They also find inverses of each transformation in the symmetry group. They then delve deeper into the algebra behind transformations, connecting them with matrix multiplication. Last, students extrapolate what they have learned about the symmetry group of a square to characterize the symmetry group of a triangle.

Teacher Preparation

This activity is designed for use in an Algebra 2 or Precalculus classroom.

- *Prior to beginning the activity, students should have experience graphing polygons in the coordinate plane and some experience with simple transformations, such as vertical and horizontal shifts. Students should also be familiar with matrix multiplication and have been introduced to the concept of inverse matrices.*
- ***To download the calculator program TRANSFOR and student worksheet, go to education.ti.com/exchange and enter "8776" in the quick search box.***

Classroom Management

- *This activity is intended to be* ***teacher-led*** *with periods of individual or small group work.*
- *If time constraints prevent you from completing the activity in one class period, you may choose to have students complete Problems 1 and 2 in class and Problem 3 as homework.*
- *The student worksheet helps to guide students through the activity, while providing a place for students to record their answers.*

TI–84 Plus Applications

none

Transformers

ID: 8776

Introduction

When a polygon is transformed, the coordinates of its vertices change. A transformation is a rule that describes how the coordinates change. Translations, reflections, rotations, and dilations are all examples of transformations.

A transformation of an object is called an isometric transformation, or isometry, if it does not change the size of the object. For example, translations are isometries, but dilations are not.

In this activity, students will explore a special subset of the isometries of a regular polygon called the symmetry group. The symmetry group is made up of all the isometries of a regular polygon whose images can be placed in the position within a "reference shape."

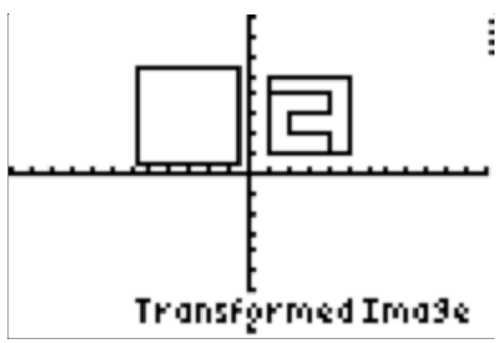
To start, have students open the **TRANSFOR** program found in the **Programs** menu.

Problem 1 – Symmetry group for a square

In this problem, students reflect and rotate a square imprinted with the letter **F** to show students how the orientation of the square changes. (The same could be accomplished by labeling vertices.)

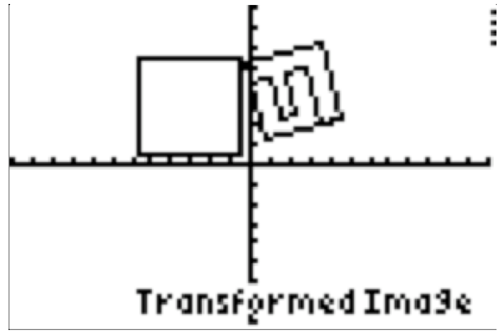
Before you begin, explain that there are an infinite number of possible transformations for any polygon. Only some of the transformations are isometries, or distance-preserving transformations. Discuss why some types of transformations are isometries and some are not, as well as how many different isometries are possible (an infinite number).

Tell students that in this activity students will explore a special subset of the isometries called the symmetry group. Transformations in the symmetry group yield images that not only preserve distance, but also "line up" with the original polygon. A reference shape is used to visually reinforce this definition.

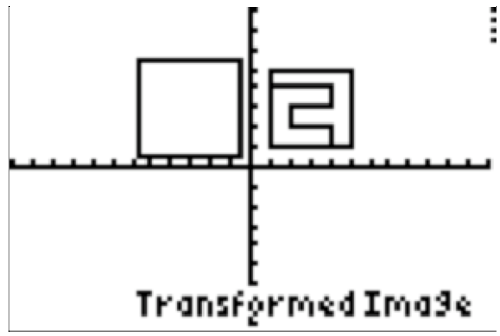


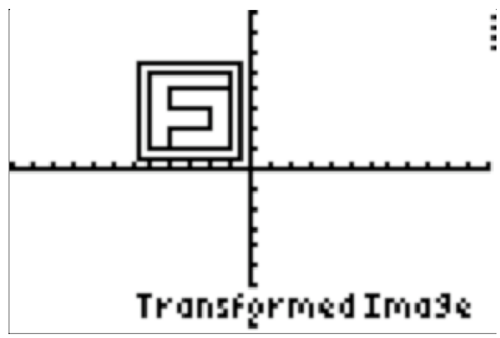
Demonstrate how to reflect the square over a line of your choosing. Run the **TRANSFOR** program and choose **Square**. To go to the menu from the Original Shape screen press ENTER, choose **Reflect**, and then the line $\boldsymbol{x = 0}$. Discuss the resulting with the class. Is it an isometry? Can it be placed in the reference square? (Does it "line up" with the original square?) Allow students to work independently or in small groups to create more reflections and determine which are in the symmetry group. They should choose **StartOver** after each transformation. The challenge of this exercise is to discover that reflections over the lines $\boldsymbol{y = x}$ and $\boldsymbol{y = -x}$ (in addition to the x- and y-axes) are also in the symmetry group. Students should record the reflections they find that are in the symmetry group on their worksheet.

Move the discussion to rotations. Demonstrate how to rotate the square. Run the **TRANSFOR** program, choose **Square**, then **Rotate**, and then enter an angle measure in degrees. (Rotation by 75°, which is not an element of the symmetry group, is shown here.) The program will rotate the shape about the origin. Allow students to work independently to find the rotations of the square that belong to the symmetry group. Again, they should choose **StartOver** after each transformation and record the rotations they find that are in the symmetry group on their worksheet.



The final step in Problem 1 is to find the inverse of each element of the symmetry group—that is, the transformation that "undoes" a transformation. This may be accomplished by figuring out how to transform a resulting image back onto the preimage. Demonstrate reflecting the square over the line *y*-axis and back again. First, reflect the square, and then choose **TransformAgain**. The calculator displays the reflected square. Explain that you are now going to transform this transformed image in an attempt to return it to its original position. Take suggestions as to which transformation might work, and then demonstrate the most popular choice.



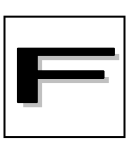


Have students work independently to find the inverses of the remaining reflections and rotations and record the results on their worksheet.

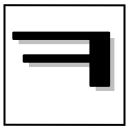
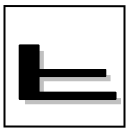
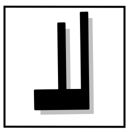
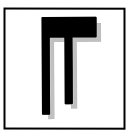
Discuss the results as a group, asking questions like: *What is the inverse of the identity transformation? How many transformations had themselves as an inverse? Were any new transformations (not in the symmetry group) needed to complete the table of inverses? Did some transformations have more than one inverse?*

Solutions

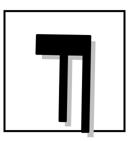
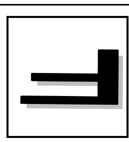
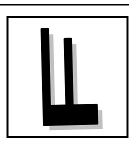
Identity

Sketch	**Description**	**Inverse**
	no change	no change

Reflections

Sketch	Description	Inverse
	reflect over $x = 0$	reflect over $x = 0$
	reflect over $y = 0$	reflect over $y = 0$
	reflect over $y = x$	reflect over $y = x$
	reflect over $y = -x$	reflect over $y = -x$

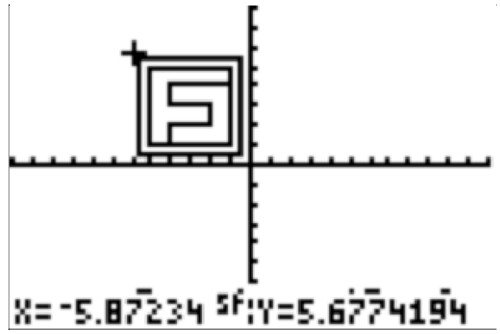
Rotations

Sketch	Description	Inverse
	rotate around origin 90° clockwise	rotate around origin 270° clockwise
	rotate around origin 180°	rotate around origin 180°
	rotate around origin 270° clockwise	rotate around origin 90° clockwise

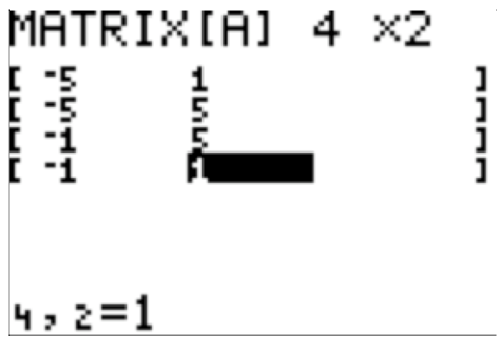
- 8
- Sample answer: the inverse transformations are all members of the symmetry group. Also, each reflection is its own inverse.

Problem 2 – Transformer matrices

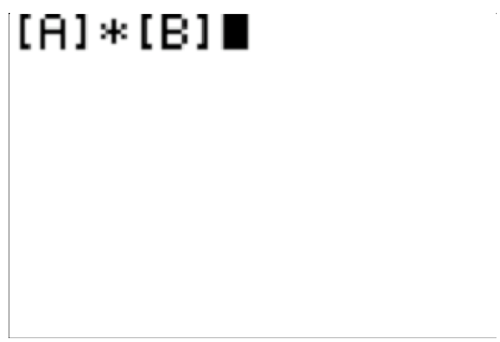
In this problem, students delve deeper into the mathematics behind these transformations. Remind them that a transformation is a rule for changing the coordinates of the vertices of a polygon. One way to record such a rule is in a matrix. Multiplying a matrix comprised of the preimage vertices by a “transformer matrix” yields a matrix comprised of the image vertices.



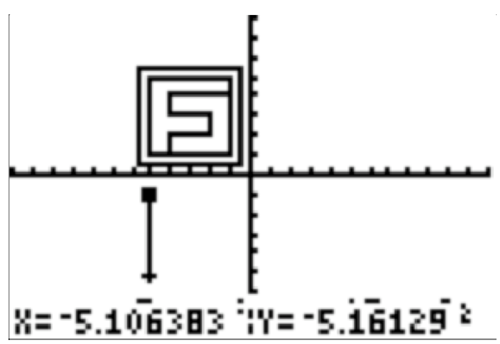
Guide students through the process of creating a matrix of vertices for the square. Run the **TRANSFOR** program. Choose **Square**, then **Back**, then **Exit**. Press $\boxed{\text{GRAPH}}$ to view the original square. Move the cursor to each corner of the square and record the coordinates, rounded to the nearest whole number.



Press $\boxed{\text{2nd}}$ $\boxed{x^{-1}}$ to open the matrix menu. Arrow over to **EDIT** and choose **[A]**. Set the dimensions to 4 × 2 and enter each vertex in a row. (The *x*-values make up the first column and the *y*-values make up the second column.) Press $\boxed{\text{2nd}}$ $\boxed{\text{MODE}}$ to exit the screen, and then edit matrix **[B]** to be the transformer matrix **T2** given on the worksheet. Have students multiply **[A]*[B]** and draw the resulting figure.

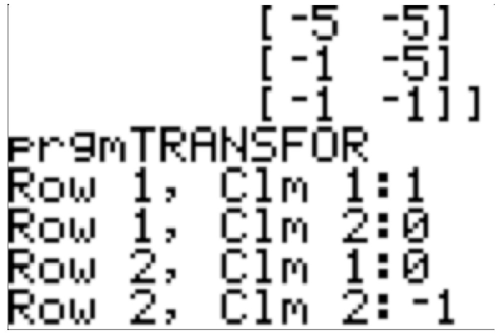


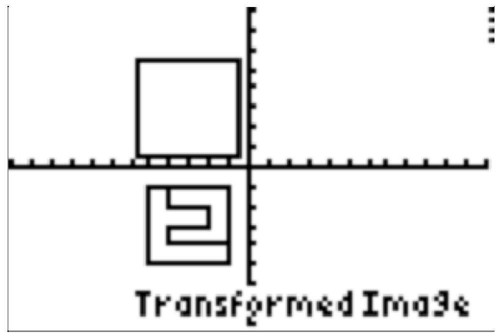
To draw the transformed square, start from the graph screen, press $\boxed{\text{2nd}}$ $\boxed{\text{PRGM}}$, and choose **Line(**. Move the cursor to the first vertex and press $\boxed{\text{ENTER}}$ to turn the pen on. Then move the cursor to the next vertex and press $\boxed{\text{ENTER}}$ to turn the pen off. Press $\boxed{\text{ENTER}}$ again to turn the pen on again and continue to draw.



Discuss the image, asking questions like: *Is this the image of a reflection? A rotation? Can you tell? Why or why not?*

To see the effect of **T2** more clearly, students should run the **TRANSFOR** program again, choose **Square** and then **Enter Matrix** and enter the elements of **T2** when prompted. The presence of the **F** makes it clear that **T2** corresponds to reflection over the line $y = 0$. Have students work independently to match each matrix on their worksheet with the transformation it represents.

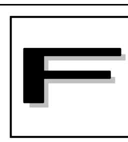
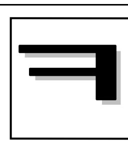
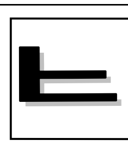


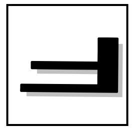
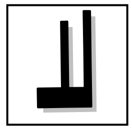
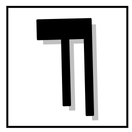
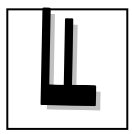
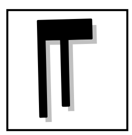


The questions on the worksheet introduce students to the idea of successive transformations. Explain that you can multiply a matrix of vertices by more than one transformer matrix to see the effect of applying more than one transformation. You can also multiply two or more transformer matrices together to create a new transformer matrix that corresponds to applying first one transformation, then the other. Students should see that multiplying the transformer matrices for transformations that are inverses yields the identity matrix.

Solutions

- $\begin{bmatrix} -7 & -7 \\ -1 & -7 \\ -1 & -1 \\ -7 & -7 \end{bmatrix}$
- reflection over $y = 0$ or rotation around origin by 270°

Transformer Matrix	Sketch	Description
$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		no change
$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$		reflect over $x = 0$
$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$		reflect over $y = 0$

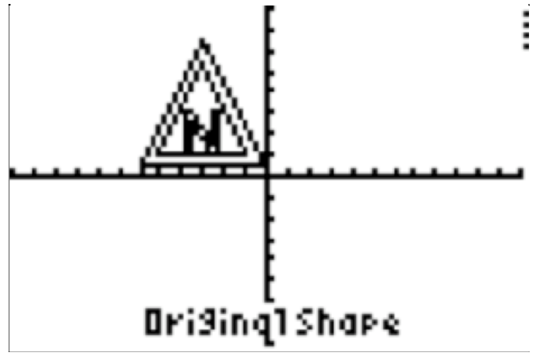
Transformer Matrix	Sketch	Description
$T_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$		rotate around origin 180°
$T_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		reflect over $y = x$
$T_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$		rotate around origin 90° clockwise
$T_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$		rotate around origin 270° clockwise
$T_7 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$		reflect over $y = -x$

Transformer Matrix	Inverse	Transformer Matrix	Inverse
$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$T_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$T_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
$T_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$T_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$T_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$T_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$T_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$T_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$T_7 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$T_7 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

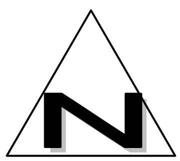
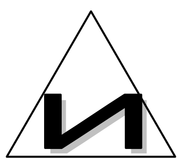
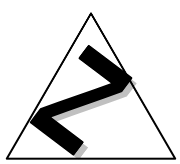
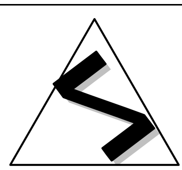
- The product of a transformer matrix and its inverse is the identity, $T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- rotating around origin 90°
- reflecting over $x = 0$

Problem 3 – Symmetry group for an equilateral triangle

In Problem 3, students use similar tools to explore a different symmetry group. Students should as before to reflect and rotate the triangle, find the inverse of each transformation, find the corresponding transformer matrices, and answer the questions on the worksheet.



Solutions

Sketch	Description	Inverse	Transformer Matrix
	identity	identity	$T_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
	reflect over $x = 0$	reflect over $x = 0$	$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
	rotate 120° clockwise	rotate 240° clockwise	$T_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$
	rotate 240° clockwise	rotate 120° clockwise	$T_3 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$