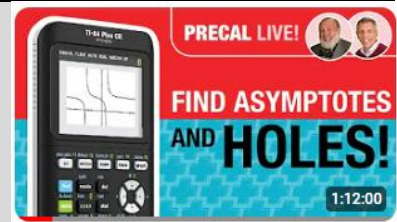





Thursday Night Precalculus Series
September 28, 2023

In this AP Precalculus Live session, we will explore several examples on finding zeros, holes, vertical asymptotes, and end behavior of rational function.



Rational Functions: Zeros, Holes, Vertical Asymptotes, and End Behavior

About the Lesson

- This Teacher Notes guide is designed to be used in conjunction with the AP Precalculus Live session and Student Problems document that can be found on-demand:
https://www.youtube.com/watch?v=8gvIFvSazDI&list=PLQa_6aWmaC6B-5h5n2Cr5h3G2ZPfJ0HGI&index=1&t=22s
 - *Please note that not all problems/content from the Student Problem Sheet is covered in the video component. Student/Teacher Notes are also useful without students viewing the “Live Session” but can be enriched by that resource.*
- This session involves exploring features of the graphs of rational functions, such as:
 - Using limits to describe end behaviors.
 - Determining the zeros.
 - Determining horizontal, slant, and vertical asymptotes.
 - Determining holes in the graphs of rational functions.
- Students should be able to use the TI-84 to verify these features of a rational function.
-  **Class Discussion:** Use these questions to help students communicate their understanding of the problem. These questions are presented in the *Live* video as well.

Materials:

- Student document*
- problems_09_28_23
- Solutions*
- problems_solutions_09_28_23
- YouTube*
- https://www.youtube.com/watch?v=8gvIFvSazDI&list=PLQa_6aWmaC6B-5h5n2Cr5h3G2ZPfJ0HGI&index=1&t=22s
 - **Documents and materials can be downloaded from this site.**

AP Precalculus Learning Objectives

- 1.7.A: Describe end behaviors of rational functions.
- 1.8.A: Determine the zeros of rational functions.
- 1.9.A: Determine asymptotes of graphs of rational functions.
- 1.10.A: Determine holes in graphs of rational functions.

Source: AP Precalculus Course and Exam Description, The College Board

Problem 1. (a)

Find the location of any zeros and holes for the graph of the given rational function.

$$f(x) = \frac{x^3 - x^2 - 10x - 8}{x + 2}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

Teacher Note: Some Algebra 2 (or equivalent) courses cover synthetic division. Synthetic division is not in the AP Precalculus curriculum and will not be assessed. Factoring polynomials without technology is limited to factoring out a common factor and the rules for quadratics.



Class Discussion:

Why does the graph look like a quadratic?

Possible Answers:

When the common factor is divided out, the remaining function in the numerator is a quadratic.

Or, the ratio of the leading terms, $\frac{x^3}{x} = x^2$ is a quadratic function.



Class Discussion:

How could you find the y -value of the hole?

Possible Answers:

To find the y -value of the hole, substitute $x = -2$ into the reduced rational function

$$f(x) = \frac{(x-4)(x+1)}{1}, x \neq -2. \text{ Hole: } (-2, 6)$$

Teacher Note: Revisit the two Class Discussion questions above when using the TI-84.

Problem 1. (b)

Find the location of any zeros and holes for the graph of the given rational function.

$$f(x) = \frac{(x+3)^2(x+1)(x-1)(x-4)}{x^2 + 2x - 3}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.



Class Discussion:

For $f(x) = \frac{(x+3)^2(x+1)(x-4)}{(x+3)(x+1)}$, would the graph be different if the factor of $(x+3)$ were

squared in the denominator instead of the numerator? $\left[g(x) = \frac{(x+3)(x+1)(x-4)}{(x+3)^2(x+1)} \right]$

Possible Answers:

The graph would now have a vertical asymptote instead of a hole at $x = -3$.

Teacher Note:

Explore the graph of $g(x)$ using technology.



Class Discussion:

How could you find the y -values of the two holes?

Possible Answers:

The hole at $x = -3$ appears to be the location where the graph is attempting to cross the x -axis. To find the y -value of the hole at $x = 1$, substitute $x = 1$ into the reduced rational function

$$f(x) = \frac{(x+3)(x+1)(x-4)}{1}, x \neq -3, 1.$$

Holes: $(-3, 0)$ $(1, -24)$

Teacher Note: Revisit the Class Discussion question above when using the TI-84.

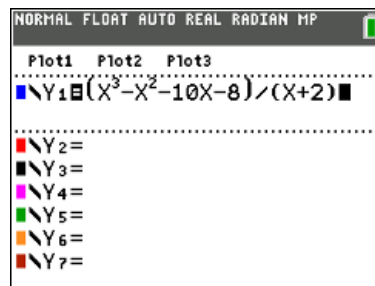
Using the TI-84 with Problem 1. (a)

Find the location of any zeros and holes for the graph of the given rational function.

$$f(x) = \frac{x^3 - x^2 - 10x - 8}{x + 2}$$

Graph the function to confirm the zeros and the location of the hole.

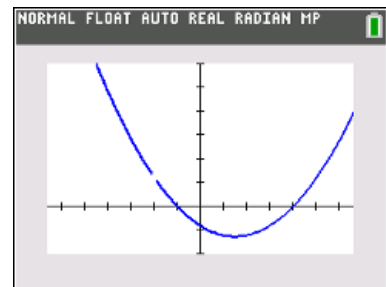
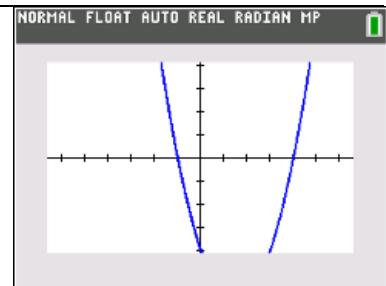
Technology Tip: Use the fraction template to enter the function instead of using parentheses. Select $\boxed{y=}$, then $\boxed{\text{math}}$ and move to the right to find the fraction template or select $\boxed{\alpha}\boxed{y=}$ to find the fraction template.



To confirm the hole at $x = -2$, use Zoom Decimal and then use Trace to move to $x = -2$.



Adjust the Window with $Y_{\min} = -10$, $Y_{\max} = 30$, and $Y_{\text{scl}} = 5$ to show the hole in the graph of the function.



Technology Tip: We could also look at a table of values of the function to confirm the locations of the hole and the intercepts. Select 2nd graph to open a table.



Class Discussion:

How could you use the table to get close to $x = -2$ in order to approximate the y -value of the hole?

What is the correct limit notation to show that the function has a hole at $x = -2$?

Possible Answers:

To change the setup of the table, select 2nd window .

Change Independent to Ask. Use the table to confirm the x-intercept.

Notice the error with $x = -2$. Use x -values close to $x = -2$. Some suggested values are $x = -1.9$ and $x = -2.1$. Also try $x = -1.99$ and $x = -2.01$. Students should observe that the y -values are getting close to 6.

$$\lim_{x \rightarrow -2} f(x) = 6$$

X	Y1			
0	-4			
1	-6			
2	-6			
3	-4			
4	0			
5	6			
6	14			
7	24			
8	36			
9	50			
10	66			

X=0

TABLE SETUP
TblStart=0
 Δ Tbl=1
Indpt: Auto Ask
Depend: Auto Ask

X	Y1			
-1	0			
-2	ERROR			
-1.9	5.31			
-2.1	6.71			
-1.99	5.9901			
-2.01	6.0701			

X=

Using the TI-84 with Problem 1. (b).

The use of the TI-84 for this problem is not discussed in the video.

Find the location of any zeros and holes for the graph of the given rational function.



$$f(x) = \frac{(x+3)^2(x+1)(x-1)(x-4)}{x^2+2x-3}$$

Use the graphing application to confirm the zeros and the hole in the graph of the function.

Try Zoom Decimal for the graph, then adjust the window settings with $Y_{\min} = -40$, $Y_{\max} = 40$, and $Y_{\text{scl}} = 10$. Use the Trace or the Table to confirm the x-intercepts and holes.



Class Discussion:

Does it make sense that the rational function looks like the graph of a cubic polynomial?

Possible Answers:

When the common factors are divided out, the remaining function in the numerator is a cubic.

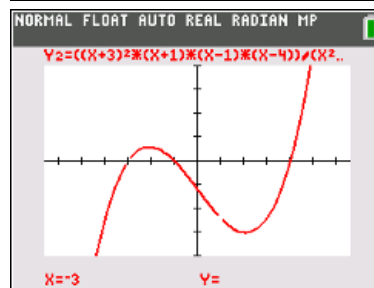
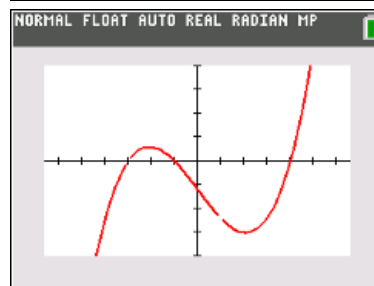
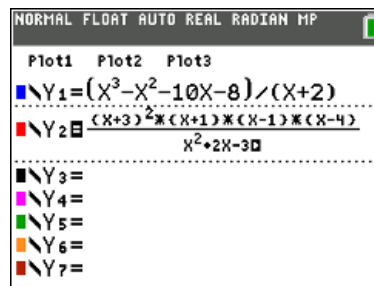
Or, the rational function $f(x) = \frac{(x+3)^2(x+1)(x-1)(x-4)}{x^2+2x-3}$ has

a polynomial of degree 5 in the numerator and degree 2 in the

denominator. The ratio of leading terms $\frac{x^5}{x^2} = x^3$ is a cubic polynomial.

Teacher Tip: Discuss why there is a hole in the graph of the function at $x = -3$. Take a look at a table of values.

Review how to find the y -value of the hole when $x = 1$.



Problem 2. (a)

Find all the vertical asymptotes for the graph of the given rational function and sketch a complete graph.

$$f(x) = \frac{x^3 - 5x^2 + 6x}{x^2 - 9}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem



Class Discussion:

What is the behavior of the function as the graph approaches the vertical asymptote, $x = -3$, on either side?

Possible Answers:

As the graph approaches -3 from the left, the y -values approach negative infinity.

$$\lim_{x \rightarrow -3^-} f(x) = -\infty$$

As the graph approaches -3 from the right, the y -values approach positive infinity.

$$\lim_{x \rightarrow -3^+} f(x) = \infty$$



Class Discussion:

What is the y -coordinate of the hole?

Possible Answers:

To find the y -value of the hole at $x = 3$, substitute $x = 3$ into the reduced rational function

$$f(x) = \frac{x(x-2)}{x+3}, x \neq -3. \quad \text{Hole: } \left(-3, \frac{1}{2}\right)$$

Problem 2. (b)

Find all the vertical asymptotes for the graph of the given rational function and sketch a complete graph.

$$f(x) = \frac{x^2 - 4}{(x-2)(x^2 - 6x + 5)}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.



Class Discussion:

Can the graph of a rational function cross its vertical asymptote?

Possible Answers:

No, it cannot.

Using the TI-84 with Problem 2. (a)

Use the graph to confirm the behavior of the graph of the function.

Use Zoom Decimal.

Use Trace to explore the zeros and the hole at $x = 3$.



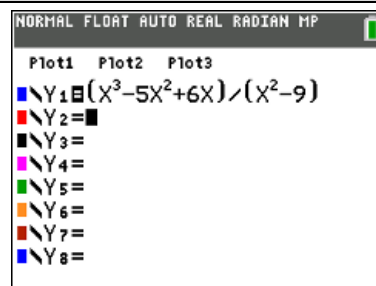
Class Discussion:

Does the graph “dip” below the x -axis?

What are the coordinates of the hole?

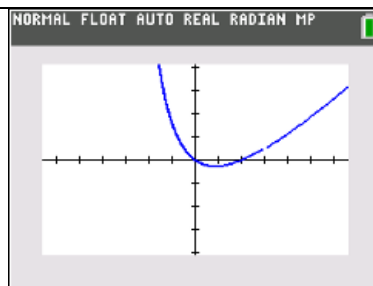
Possible Answers:

Yes, the graph dips below the x -axis.





Use Trace to explore values of the function near $x = 3$. We could also use the table or find the coordinate analytically. The y-coordinate of the hole is $\frac{1}{2}$. Using a limit, we would write $\lim_{x \rightarrow 3} f(x) = \frac{1}{2}$.

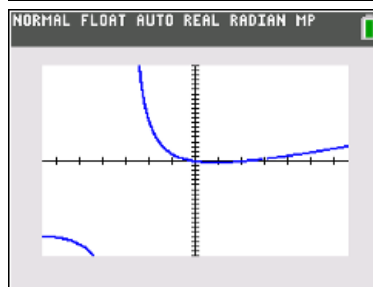


To observe more features of the graph, adjust the window settings with Ymin = -20 and Ymax = 20.



Class Discussion:

How do we use limit notation to describe the behavior of the function as the graph approaches -3 from both the left and the right?



Possible Answers:

$$\lim_{x \rightarrow -3^-} f(x) = -\infty$$

$$\lim_{x \rightarrow -3^+} f(x) = \infty$$

Using the TI-84 with Problem 2. (b)

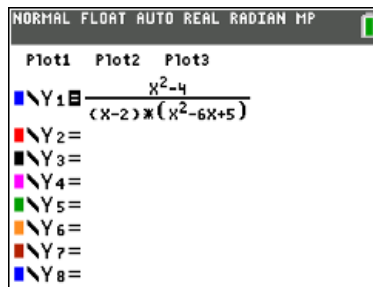
The use of the TI-84 for this problem is not discussed in the video.

Use the graph to confirm the behavior of the graph of the function near the vertical asymptotes.



Class Discussion:

Use limit notation to describe the behavior of the graph on either side of the vertical asymptotes.



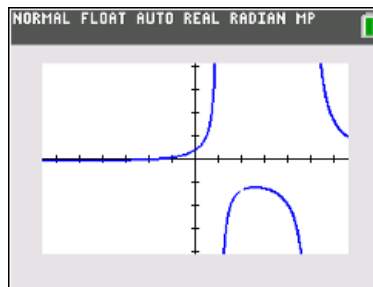
Possible Answers:

$$\lim_{x \rightarrow 1^-} f(x) = \infty$$

$$\lim_{x \rightarrow 5^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 1^+} f(x) = -\infty$$

$$\lim_{x \rightarrow 5^+} f(x) = \infty$$



Using the Transformation App on the TI-84

Here is a Youtube Video in the Texas Instruments Education Channel on using the Transformation App.

<https://www.youtube.com/watch?v=WoBfrTDx2LQ>



Technology Tip: Select [apps] then select Transfrm.

Select [y=] and notice that Y_1 and Y_2 have different symbols and can both be used for transformations.

Select [y=] and type $y_1 = \frac{(x-A)(x-B)}{(x-C)(x-D)}$. A , B , C , and D are

constants that will be used as parameters.

Teacher Tip: Use the fraction template instead of the parentheses shown in the video.

Graph using Zoom Decimal.



Class Discussion:

When $A=1$, $B=1$, $C=1$, and $D=1$, why does the graph appear to be the horizontal line $y=1$?

Possible Answers:

The function is not defined at $x=1$. The numerator and denominator are exactly the same, so the function is equivalent to 1 with $x \neq 1$.



Class Discussion:

If $A=2$, $B=1$, $C=1$, and $D=1$, what do you notice about the graph of the function?

Possible Answers:

The function has a zero at $x=2$. We have three linear factors of $x-1$, one in the numerator and two in the denominator. Because we have more factors of $x-1$ in the denominator than the numerator, we have a vertical asymptote $x=1$.



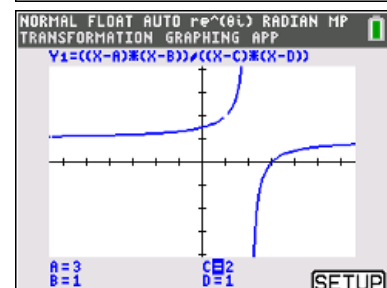
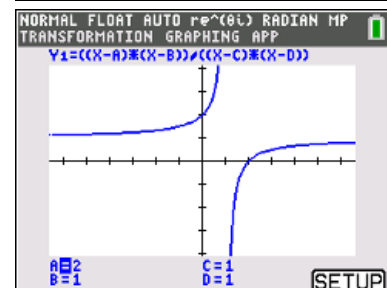
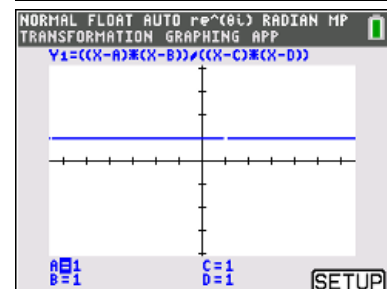
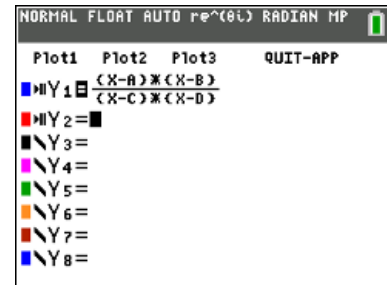
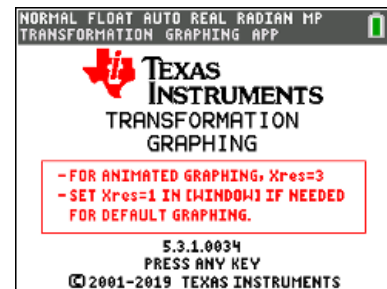
Class Discussion:

How could we change the values of A , B , C , and D to have a hole at $x=1$?

Possible Answers:

One possibility is $A=3$, $B=1$, $C=2$, and $D=1$.

Teacher Tip: The transformation app allows the teacher to “jeopardize” questions in class. Challenge the students to create a graph with certain characteristics.





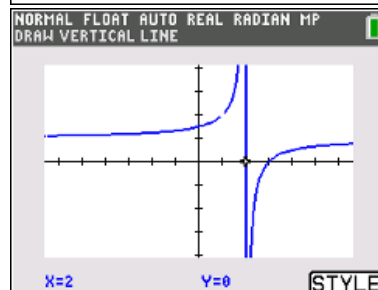
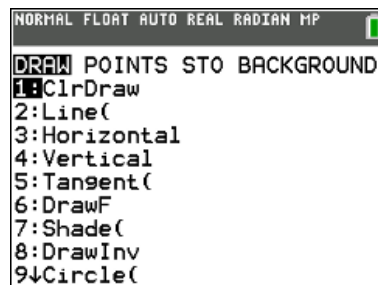
Class Discussion:

How could we change the values of A , B , C , and D to have a zero at $x = 1$, a hole at $x = 2$, and a vertical asymptote at $x = 3$?

Possible Answers:

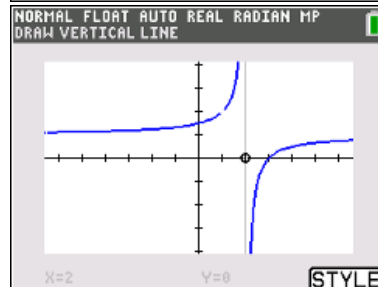
One possibility is $A = 1$, $B = 2$, $C = 2$, and $D = 3$. $f(x) = \frac{(x-1)(x-2)}{(x-2)(x-3)}$.

Technology Tip: Select $\boxed{2\text{nd}}\boxed{\text{prgm}}$ to draw horizontal and vertical lines to represent asymptotes.



To activate the “soft” key Style, select $\boxed{\text{graph}}$.

Try setting the color to Medium Gray and the line style to thin.



Note: The following problems are not discussed in the video.

Problem 3. (a)

Express the end behavior of each rational function using limit notation and sketch a complete graph.

$$f(x) = \frac{10x}{x^2 + 4}$$

Sample Solution:



Refer to the Teacher Solutions Document for the full solution to this problem.

Problem 3. (b)

Express the end behavior of each rational function using limit notation and sketch a complete graph.

$$f(x) = \frac{3x^2 - 15}{x^2 - 2x - 8}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

Problem 3. (c)

Express the end behavior of each rational function using limit notation and sketch a complete graph.

$$f(x) = \frac{x^3 - x}{x^2 - 4}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- The graph of the function can be used to explore zeros and holes.
- The graph of the function can be used to explore the behavior of a function near the vertical, horizontal, and slant asymptotes.
- The use of tables on the calculator also supports conclusions about the behavior of rational functions.

For more videos from the AP Precalculus Live series, visit our playlist

https://www.youtube.com/playlist?list=PLQa_6aWmaC6B-5h5n2Cr5h3G2ZPfJ0HGI

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