Hourglass

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Abstract: This activity is an application of differentiation and integration. Students are asked to consider the rate of flow in an hourglass. They first compute the volume of part of the hourglass and use this to relate the flow of sand to the change in the height of the change. They then use the symbolic capacity of their calculator and calculus to determine the exact rate of change.

NCTM Principles and Standards:
Algebra standards
a) Understand patterns, relations, and functions
b) generalize patterns using explicitly defined and recursively defined functions;
c) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
d) use symbolic algebra to represent and explain mathematical relationships;
e) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
f) draw reasonable conclusions about a situation being modeled.

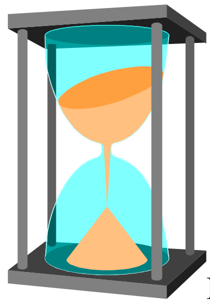
Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard
a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

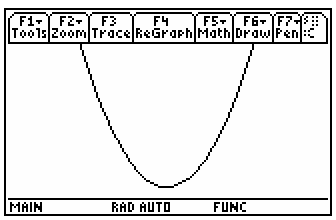
Representation Standard : use representations to model and interpret physical, social, and phenomena.

Key topic: Applications of Derivative - Related Rates: determining the rate of change, Application of Integration - finding volume of rotation.

Degree of Difficulty: Elementary to moderate
Needed Materials: TI-89 calculator

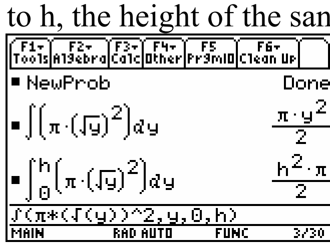
Situation: Consider an hour glass:  Let's assume that the sand is falling at the rate of 2 cc/second. When the height of the sand is 5 cm, at what rate is the height changing? The halves of the hourglass are paraboloids as shown in the diagram. The generating equation of the parabola is $y = x^2$.

We first need to find the volume of the volume of the paraboloids in terms of their height - which is the distance from the vertex.



To find the volume of the solid generated by this curve as it is revolved about its axis of symmetry, use the disk method.

A typical cross section has a radius of $x = \sqrt{y}$ and a thickness of Δy. We want to find the volume of this solid from 0 to h, the height of the sand. Therefore our integral representing the volume is:



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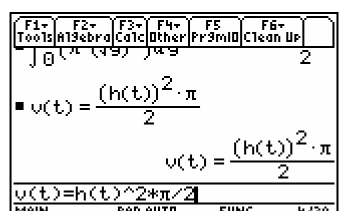
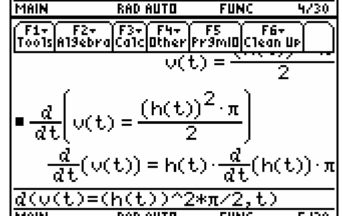
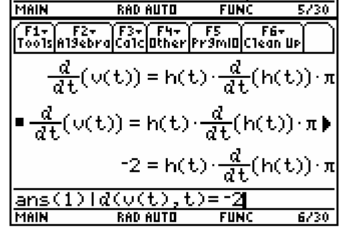
Both V and h are functions of time, so we need to reflect that fact when we enter the equation in the calculator:



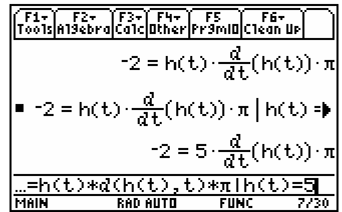
Now take the derivative of both sides with respect to time, t.



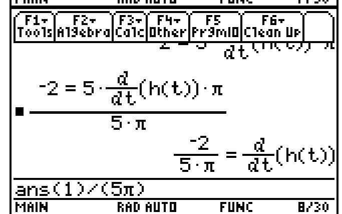
We know that $\frac{dv}{dt} = -2$ and we can use that fact to help solve for $\frac{dh}{dt}$



We also know that we are evaluating this when the height = 5 cm.



We can solve for $\frac{dh}{dt}$



By pressing ◆ we can evaluate this expression: