NUMB3RS Activity: Harmonizing Means
Episode: “Longshot”

Topic: Harmonic mean **Grade Level:** 9 - 12
Objective: Explore the harmonic mean and its applications; compare the harmonic mean to the arithmetic and geometric means.
Materials: Scientific or graphing calculator
Time: 15 - 20 minutes

Introduction
In “Longshot,” a man is murdered at a horse racetrack after he wins a large sum of money. The FBI shows Charlie the man’s notebook that contains a lot of horse racing data and equations. Some of these equations involve the harmonic mean. The harmonic mean is one of several methods of calculating *an* average. The harmonic mean of a group of terms is the number of terms divided by the sum of the terms’ reciprocals. Typically, it is appropriate for situations involving the average of rates.

Discuss with Students
This activity provides a good opportunity to review mean, median, and mode, as well as the appropriateness of using each of these to describe characteristics of a set of numbers. Remind students that “arithmetic mean” is the same as the “average.” The geometric mean is used extensively in geometry. The geometric mean of two numbers is called the mean proportion of the numbers.

Student Page Answers:
1*.* *Greatest: arithmetic; least: harmonic.* **2.** *Answers vary, but generally the arithmetic mean will be the greatest and the harmonic mean the least, as long as the numbers aren’t all equal.*
3. *If the three numbers are equal, the three means will also be equal.* **4a.** *A: 8 mph, B: 12 mph* and *10 mph, 9.6 mph, and about 9.8 mph* **4b.** *This value is the same as the arithmetic mean.*
4c. *C: 20 mph, D: 30 mph; 25 mph, 24 mph, and about 24.49 mph* **4d.** *This value is the same as the harmonic mean.* **4e.** *E: 12 mph, F: 16 mph. On average the buses travel at 14.4 mph (12 miles in 50 minutes.) Arithmetic: 14 mph, Harmonic: 13.71 mph, Geometric: 13.86 mph. In this situation, none of the three means gives the correct answer.* **5.** *Initially: 18.18 mpg, tuning up the SUV: 19.82 mpg, replacing the hybrid: 19.04 mpg. The couple should tune the SUV.* **6.** *The harmonic mean is $9,600. This is 20% more than the buyer wanted to pay and 20% less than the seller wanted to get.* **7.** *80.3%, 79.4%, 79.9%. These methods might all be fair, but students are more likely to prefer the use of the arithmetic mean as it results in higher averages.*

Extensions Answers: 1. *Harmonic mean < geometric mean < arithmetic mean.* **2.** *FG = 6, the harmonic mean of 4 and 12. Hint: Use similar triangles and the theorem that parallel lines cut off proportional segments on transversals to find equations for FE and EG in terms of the bases of the trapezoid.*

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Name: ___________________________________ Date: ________________

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In "Longshot," the FBI enlists Charlie's help after a man is murdered at a horse racetrack. The man had won the Pick 6 (picking the winner of six consecutive races) and was due a large sum of money. The FBI shows Charlie the man's notebook that contains a lot of horse racing data and equations. Charlie determines that the equations were designed to pick the second place horse, but not the first place horse. Charlie observes that some of these equations involve the harmonic mean.

The word "mean" is used in mathematics to describe the average of a set of a numbers. The formal name for average is "arithmetic mean." There are other means, including the harmonic mean and the geometric mean. Each of them has many uses, and each is a measure of the center of a set of numbers. In a sense, each of these means represents a value that is "typical" of the set of numbers.

The formulas for computing these means for positive numbers are shown below. These formulas can be extended to four or more numbers.

	Two numbers: *a* and *b*	Three numbers: *a*, *b*, and *c*
Arithmetic Mean	$\frac{a+b}{2}$	$\frac{a+b+c}{3}$
Harmonic Mean	$\frac{2}{\left(\frac{1}{a}+\frac{1}{b}\right)}$	$\frac{3}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$
Geometric Mean	$\sqrt{a \cdot b}$	$\sqrt[3]{a \cdot b \cdot c}$

Example: Compute the arithmetic, harmonic, and geometric means of the numbers 8, 12, and 66.

Arithmetic Mean	$\frac{8+12+66}{3} \approx 28.7$
Harmonic Mean	$\frac{3}{\left(\frac{1}{8}+\frac{1}{12}+\frac{1}{66}\right)} \approx 13.4$
Geometric Mean	$\sqrt[3]{8 \cdot 12 \cdot 66} \approx 18.5$

1. In the above example, which of the means is the greatest? The least?

2. Pick three different numbers and compute their arithmetic, harmonic, and geometric means. Which is the greatest? The least?

3. Find two numbers whose arithmetic mean is the same as their harmonic mean. How do their geometric means compare? Find three numbers that have this same property.

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4. a. Suppose student A lives 4 miles from school and student B lives 6 miles from school. The two students each take a different bus to school and each bus ride takes 30 minutes. Find each bus's rate in miles per hour. Then find the arithmetic, harmonic, and geometric means of the bus's rates.

b. These two buses travel a total of 10 miles in a half hour. Therefore, on average, they travel at the rate of 10 miles per hour. Is this the arithmetic, harmonic, or geometric mean of the two bus's rates?

c. Two other students live in different places, each 10 miles from school. Student C's bus ride is 30 minutes long and student D's bus ride is 20 minutes long. Find each bus's rate in miles per hour. Then find the arithmetic, harmonic, and geometric means of the bus's rates.

d. The buses travel a total of 20 miles in 50 minutes. Therefore, on average, they travel at the rate of 24 miles per hour. Is this the arithmetic, harmonic, or geometric mean of the two bus's rates?

e. The arithmetic and harmonic means can often be used to calculate average rates. In Question 4b, the buses traveled for the same time. In Question 4d, the buses traveled the same distance. Consider the situation where two buses travel for different lengths of time and different distances. Student E's bus travels 4 miles in 20 minutes and student F's bus travels 8 miles in 30 minutes. Thus, the two buses traveled a total of 12 miles in 50 minutes. Compare this rate to the arithmetic, harmonic, and geometric means of the individual bus's rates.

A few years ago, the NPR radio program "Car Talk" featured a problem involving a hypothetical couple who had two vehicles: a gas-guzzling SUV, and a super efficient hybrid. The SUV got 10 miles per gallon (mpg) and the hybrid got 100 mpg. Each car was driven the same number of miles each year. The couple wanted to improve the average number of miles per gallon for their household. They had two choices: tune up the SUV so that it would get 11 mpg, or replace the hybrid with a "super-duper hybrid" which would get 200 mpg.

5. Use the harmonic mean to compute the initial average miles per gallon of the household and for both of their choices. What would you advise the couple to do?

Xenophon, the Greek scholar (c. 427 – 355 B.C.), used the harmonic mean to establish a fair transaction price. Here is a modern example: Suppose you wish to buy a used car. You feel that the car is only worth $8,000. The seller feels that the car is worth $12,000. A neutral observer suggests that you "split the difference" and agree to a price of $10,000 (the arithmetic mean of the two values). You argue that this value is 25% more than what you think the car is worth, but only 16.7% less than seller's figure, and therefore using the arithmetic mean isn't fair to you. The seller says, "Okay, if you show me a fairer method, I'll make the deal."

6. Use what you've learned in this activity to find a fairer price, and explain how you would convince the seller that this new price is fair. Calculate the harmonic mean of your price and the seller's price. Why would using the harmonic mean be fairer when "splitting the difference" than using the arithmetic mean?

A math teacher wants to give a grade to her students for a two-week period. Suppose that during these two weeks a student earned 16 of 20 points on one assignment, 21 of 30 points on another assignment, and 91 out of 100 points on a quiz. Each score is worth the same amount toward the final grade. The teacher is considering computing the grades by using arithmetic, harmonic, or geometric means.

7. What would the student's grade be under each system? Would each of these methods be fair? Which method would the student hope the teacher chooses? Why?

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

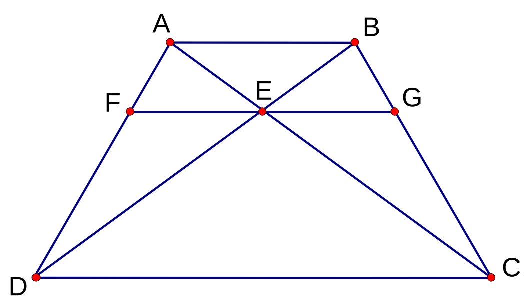
1. **a.** Based on your work in Questions 1 and 2, make a conjecture about the ranking, from least to greatest, of the arithmetic, harmonic, and geometric means of two different positive numbers.

b. Start a proof of your conjecture by first showing that the greatest mean is greater than the one in the middle.

c. Complete your proof by proving that the middle mean is greater than the least one.

2. The NCTM "Student Math Notes" (November 2004) featured the following problem:

Let *ABCD* be an isosceles trapezoid with *AB* = 4 and *DC* = 12. Find the length of $\overline{FG}$, the line segment through the intersection of the diagonals, *E*, that is parallel to the bases. How does this value relate to the arithmetic, harmonic, or geometric mean of 4 and 12?



3. The harmonic mean was proposed as a rounding method for apportioning the US House of Representatives in 1820 by James Dean. The current method uses the geometric mean. For more information, see **http://www.siam.org/pdf/news/552.pdf**.

Additional Resource

This article contains activities that focus on how to decide whether to use the harmonic or arithmetic mean to solve a particular problem:

S. L. Brown and M. A. Rizzardi, "Averaging Rates: Deciding When to Use the Harmonic or Arithmetic Mean" Mathematics Teacher, Vol. 98, No. 9, May, 2005.

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