Cyclic Quadrilaterals

Definition:

Cyclic quadrilateral—a quadrilateral inscribed in a circle (Figure 1).

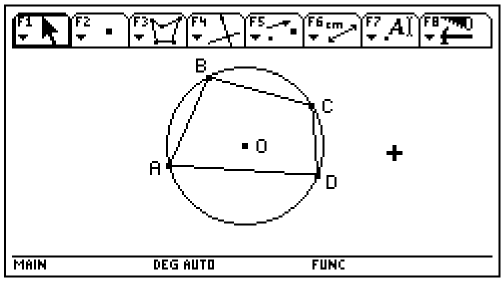

Figure 1

Construct and Investigate:

1. Construct a circle on the Voyage™ 200 with Cabri screen, and label its center ***O***. Using the **Polygon** tool, construct quadrilateral ***ABCD*** where ***A***, ***B***, ***C***, and ***D*** are on circle ***O***. By the definition given above, ***ABCD*** is a cyclic quadrilateral (Figure 1).
   Cyclic quadrilaterals have many interesting and surprising properties. Use the Voyage 200 with Cabri tools to investigate the properties of cyclic quadrilateral ***ABCD***. See whether you can discover several relationships that appear to be true regardless of the size of the circle or the location of ***A***, ***B***, ***C***, and ***D*** on the circle.
2. Measure the lengths of the sides and diagonals of quadrilateral ***ABCD***. See whether you can discover a relationship that is always true of these six measurements for all cyclic quadrilaterals. This relationship has been known for 1800 years and is called **Ptolemy's Theorem** after Alexandrian mathematician Claudius Ptolemaeus (A.D. 85 to 165).
3. Determine which quadrilaterals from the quadrilateral hierarchy can be cyclic quadrilaterals (Figure 2).

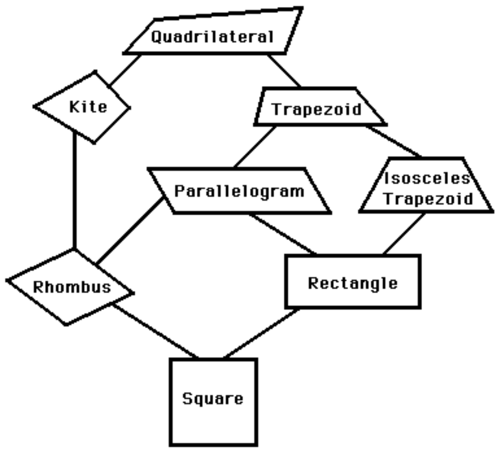

Figure 2

4. Over 1300 years ago, the Hindu mathematician Brahmagupta discovered that the area of a cyclic quadrilateral can be determined by the formula:
   $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where a, b, c, and d are the lengths of the sides of the quadrilateral and s is the semiperimeter given by $s = \frac{a+b+c+d}{2}$.
   Using cyclic quadrilaterals, verify these relationships. What should you find if $d = 0$? What is the name of this formula, and when was it first discovered?
5. Construct any quadrilateral. Show that the internal angle bisectors are always concurrent or intersect in four points that are the vertices of a cyclic quadrilateral.

Explore:

1. Assume ***P*** is the intersection point of the diagonals of quadrilateral ***ABCD***. Investigate relationships between segments $\overline{AP}$, $\overline{BP}$, $\overline{CP}$, and $\overline{DP}$ that are true when ***ABCD*** is a cyclic quadrilateral but not true when ***ABCD*** is a noncyclic quadrilateral.
2. Construct a cyclic quadrilateral. Pick any point ***P*** on the circumcircle of the quadrilateral. Find the perpendicular distance from this point to each of the sides (or extensions of the sides) of the quadrilateral and to the two diagonals. What relationship exists among these distances regardless of the location of ***P*** on the circle?

Construct and Investigate:

1. Students might discover the following relationships:

   - Opposite interior angles are supplementary (Figure 3).

   - The perpendicular bisectors of the sides of the quadrilateral are concurrent at a point if, and only if, the quadrilateral is cyclic. The perpendicular bisectors of quadrilateral ***ABCD*** shown in Figure 4 are concurrent at ***O***, the center of the circumcircle. This is true because the sides of a cyclic quadrilateral are chords of the circumcircle. The perpendicular bisector of a chord of a circle passes through the center of the circle.

   - The **altitude** of a triangle is a segment from a vertex perpendicular to the opposite side or an extension of the opposite side of the triangle. The analogous construction in a quadrilateral is a **maltitude**. A maltitude is a segment from the midpoint of a side perpendicular to the opposite side of a quadrilateral. The maltitudes of a cyclic quadrilateral are concurrent at a point (Figure 5).

   - Assume you have four segments of unequal lengths in which the sum of all four lengths is less than the sum of any three. You can form three different cyclic quadrilaterals that have the *maximum* area possible for the four given lengths (Figure 6).

     For more information on this construction, see *Maximizing the Area of a Quadrilateral* in *Explorations for the Mathematics Classroom* (Vonder Embse and Engebretsen, 1994, p. 44).

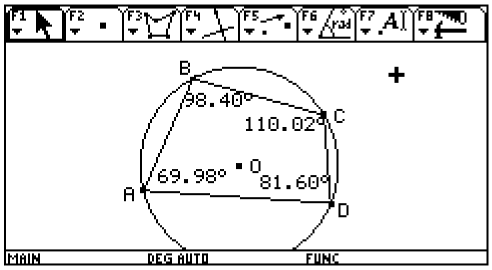

Figure 3

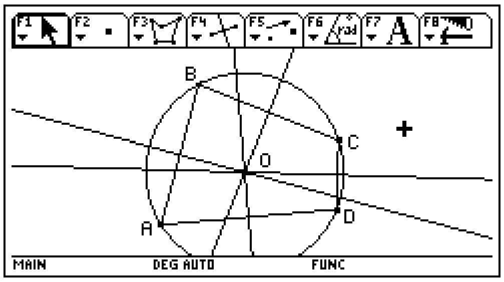

Figure 4

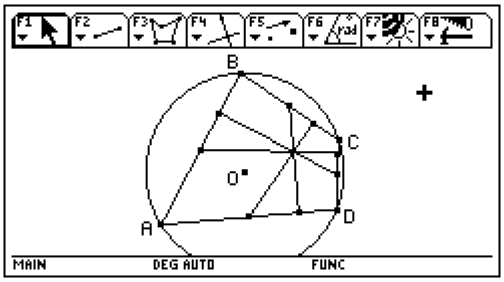

Figure 5

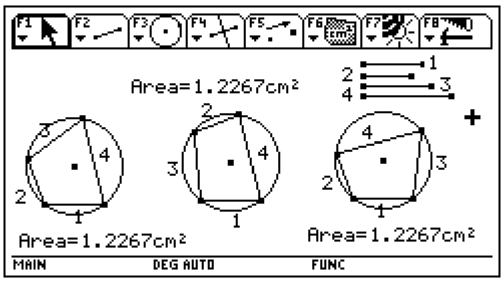

Figure 6

- A continuous path in one circuit is always possible for a ball bouncing inside a cyclic quadrilateral if the center of the circumcircle is inside the quadrilateral. The continuous path that the ball follows is the shortest path for the ball such that it touches each side of the quadrilateral and returns to its original position. At each side of the quadrilateral, the angle of incidence is equal to the angle of reflection.

  The path can be constructed by reflecting point $\boldsymbol{P}$ and its reflections $\boldsymbol{P}'$, $\boldsymbol{P}''$, and $\boldsymbol{P}'''$ over successive sides of the cyclic quadrilateral to locate point $\boldsymbol{P}''''$.

  Construct segment $\overline{\boldsymbol{PP}''''}$. Connect the point of intersection of this segment and the side of the cyclic quadrilateral with point $\boldsymbol{P}''''$.

  Connect the point of intersection of this segment to $\boldsymbol{P}''$ and so forth until you get back to $\boldsymbol{P}$ (Figure 7).

  The polygon containing the four points of intersection passes through point $\boldsymbol{P}$. This polygon represents the continuous path of the ball bouncing inside the cyclic quadrilateral (Figure 8).

  For more information on this construction, see *The Billiards Table Problem* in *Explorations for the Mathematics Classroom* (Vonder Embse and Engebretsen, 1994, p. 40).

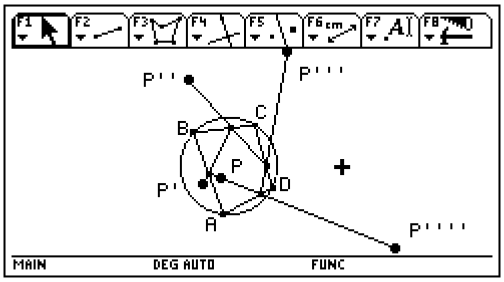

Figure 7

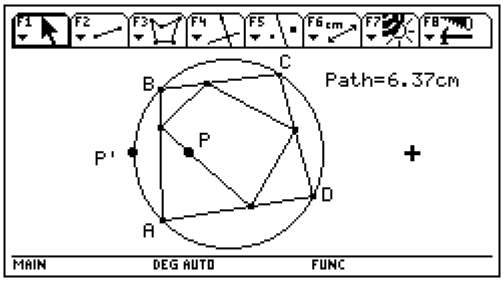

Figure 8

2. Ptolemy's Theorem states that if a quadrilateral is inscribed in a circle, then the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.

   For cyclic quadrilateral ***ABCD*** shown in Figures 9 and 10, Ptolemy's Theorem states:

   $\boldsymbol{AB} * \boldsymbol{CD} + \boldsymbol{BC} * \boldsymbol{AD} = \boldsymbol{AC} * \boldsymbol{BD}$.

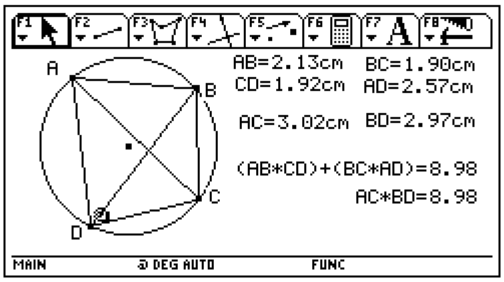

Figure 9

Drag the vertices of this quadrilateral around the circle to show that Ptolemy's relationship is always true as long as the figure remains a quadrilateral (Figure 10).

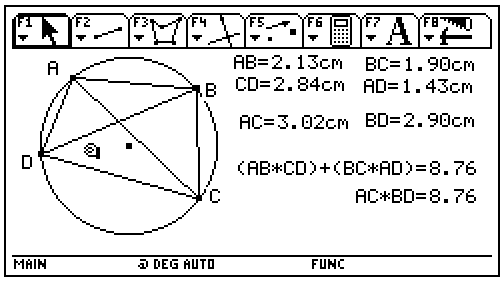

Figure 10

Use the **Redefine Point** tool to free point ***D*** from the circle. It is interesting to note that if ***D*** is not on the circumcircle that contains points ***A***, ***B***, and ***C***, then the relationship changes as follows:

AB * ***DC*** + ***AD*** * ***CB*** > ***AC*** ****BD***.

This inequality is the same if point ***D*** is inside or outside the circle (Figures 11 and 12).

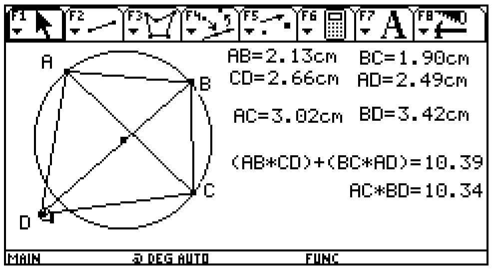

Figure 11

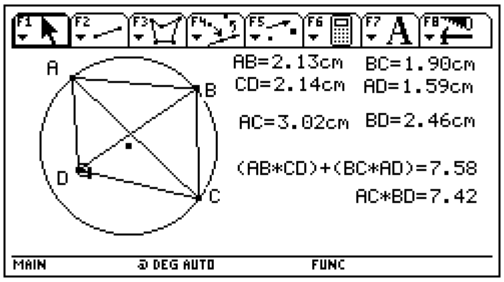

Figure 12

3. If a trapezoid is cyclic, then it must be isosceles. Two parallel chords of a circle subtend equal arcs between their endpoints. These equal arcs have equal chords that are the nonparallel sides of the trapezoid. Therefore, a cyclic trapezoid must be an isosceles trapezoid (Figure 13).

   If a kite is cyclic, then its line of symmetry is the diameter of the circle and the nonvertex angles are right.

   If a parallelogram is cyclic, then it must be a rectangle. The perpendicular bisectors of the sides of a parallelogram are concurrent only when it is a rectangle. For a parallelogram to be cyclic, its perpendicular bisectors must be concurrent at the center of the circumcircle; therefore, a cyclic parallelogram must be a rectangle (Figure 14).

   If a rhombus is cyclic, then it must be a square. The proof of this fact follows the same reasoning as for a general parallelogram.

4. If d = 0 in Brahmagupta's formula, then the formula $A = \sqrt{s\,(s - a)\,(s - b)\,(s - c)}$ can be used to find the area of any triangle with sides a, b, and c and semi-perimeter s. This equation is called **Heron's formula** after Heron of Alexandria (approximately A.D. 60). Some historians claim that the formula was known by Archimedes in the third century B.C. For more information, see *A History of Greek Mathematics*, *Vol. 2* (Heath, 1991, pp. 298–354).

5. Figure 15 shows the four internal angle bisectors defining a cyclic quadrilateral.

   Figure 16 shows the internal angle bisectors concurrent at a point.

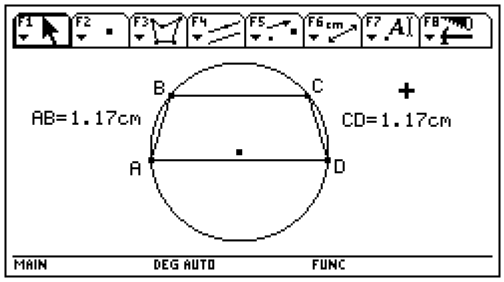

Figure 13

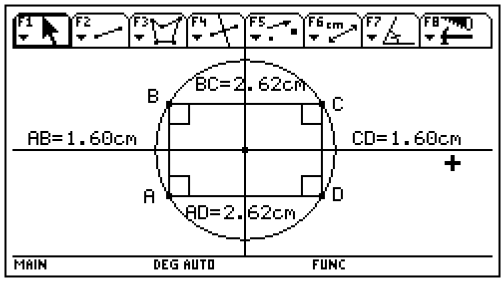

Figure 14

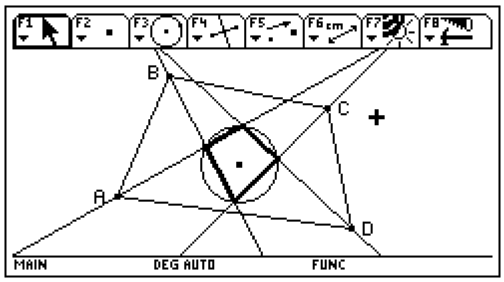

Figure 15

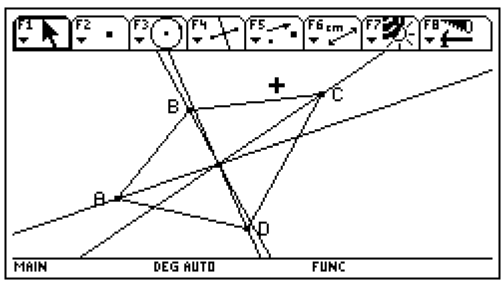

Figure 16

Explore:

1. If ***P*** is the intersection of the diagonals of a cyclic quadrilateral ***ABCD***, then:

   AP* * *CP = ***BP* * *DP*** (Figure 17).

   This property is true only for cyclic quadrilaterals. Use **Redefine Point** to free a point from the circle, and then move it to show that the products given above are no longer equal (Figure 18).

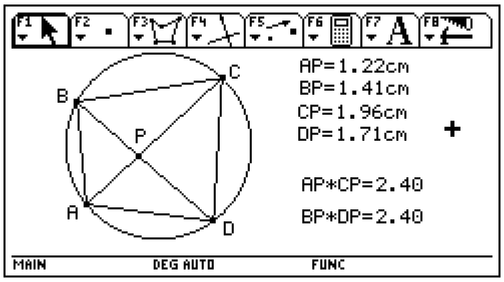

Figure 17

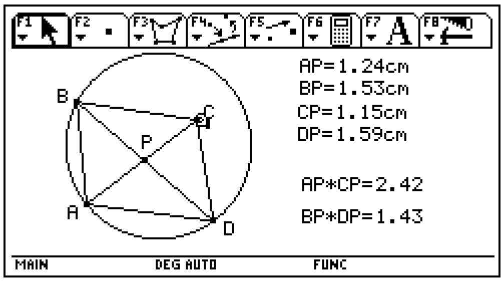

Figure 18

2. In Figures 19 and 20, points ***G*** and ***H*** are the intersection points of the perpendicular lines through point ***P*** and the diagonals of the quadrilateral. Points ***K***, ***L***, ***M***, and ***N*** are the intersections of perpendicular lines through point ***P*** and the four sides of the quadrilaterals or extensions of the sides.

   The product of the distances from ***P*** to two opposite sides of the quadrilateral is equal to the product of the distances from ***P*** to the *other* two opposite sides of the quadrilateral. This product is equal to the product of the distances from ***P*** to the diagonals of the quadrilateral.

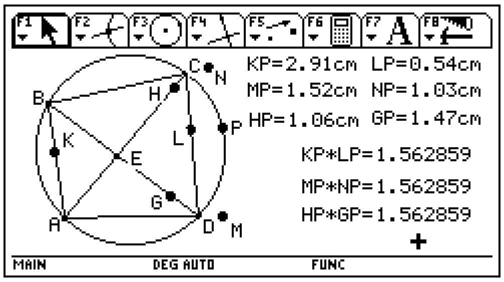

Figure 19

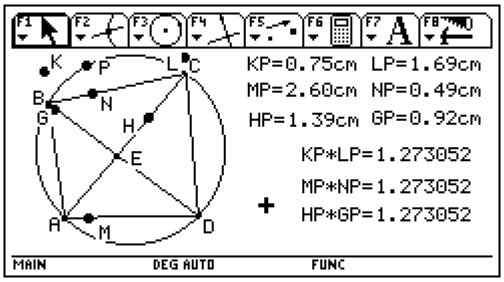

Figure 20

