Linear Approximation

LinearApprox.tns

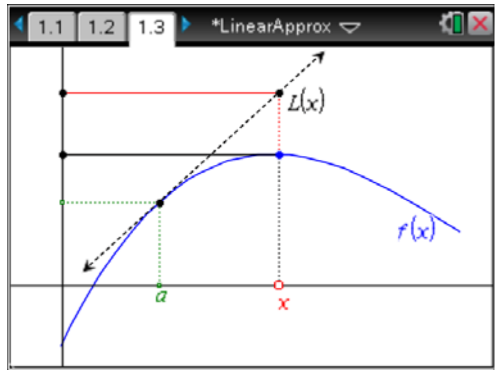
Name ______________________

Class ______________________

Introduction

The main concept in this activity, linear approximation, relates to the fact that you can use a tangent line to estimate values of a function near the point of tangency. For this reason, *linear approximation* is also sometimes referred to as *tangent line approximation*.

The diagram on page 1.3 shows a tangent line, $L(x)$, drawn to a function $f(x)$ at $x = a$.



- Observe the three points on the *y*-axis. Using *a*, *x*, and function notation, determine the appropriate labels for these points and mark them on the diagram to the right.

- Which of these three labels do you think can be used to represent the estimate, or linear approximation, of $f(x)$ near *a*?

- How can we use these labels to represent the *error* associated with this estimate?

- Is this estimate an *overestimate* or an *underestimate*? Explain.

Investigating linear approximation

On page 1.4, the graph of $f(x) = x^3 - 3x^2 - 2x + 6$ is shown and the tangent to $f(x)$ at $a = -1$.

- Which numerical value on the screen represents the linear approximation of $f(x)$ near $a = -1$?

- Which numerical value on the screen gives the error associated with this linear approximation?

- Which numerical value on the screen gives the *true* value of $f(x)$ associated with this linear approximation?

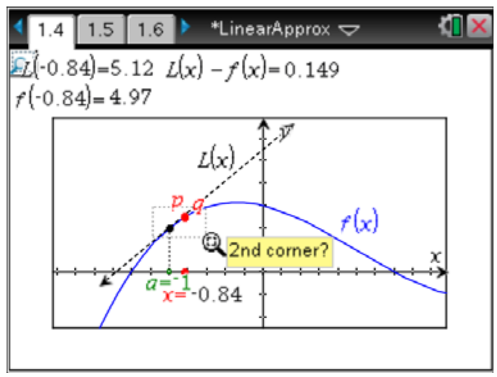
- Is this linear approximation an *overestimate* or an *underestimate*? Explain.

- Drag point *p* slowly towards the point of tangency. As you do so, what happens to the error associated with the linear approximation?

- Continue dragging point *p* so that the associated error becomes less than 0.5. What is the value of the approximation? How close to –1 does point *p* need to be such that the error becomes less than 0.5?

- Now drag point *p* to find and record the approximation and error for a value of *x* less than –1. Is this approximation an *overestimate* or an *underestimate*?

Zooming in on the graph can give you a better view of the linear approximation. Your goal is to create a box around the point of tangency and that includes points *p* and *q*. Select **Zoom – Box** from the Window menu and you will be prompted for the "1st corner." Move your cursor to a location marking the upper left corner of your box and press [enter]. Then move to the position you want for the lower right corner and press [enter] again. The graph will be magnified around the specified area. You can continue to use **Zoom – Box** to gain more magnified views or press [esc] to again drag point *p*.



- What do you notice about the appearance of the graph as you zoom in on the point of tangency?

- Based on your observations, explain why the relationship between a tangent and a graph at the point of tangency is often referred to as *local linearization*.

Typically, you do not have the benefit of a graph to find a linear approximation. Rather, you must determine the equation of the tangent line and use this equation to estimate function values near a point of tangency.

- Find the equation of the tangent line to $f(x) = x^3 - 3x^2 - 2x + 6$ at $x = -1$ and use the *Calculator* application on page 1.5 to define it as function $L(x)$.

  $L(x) =$ ____________

- What is $L(-1.03)$? What does this value represent?

- Calculate the error associated with this estimate.

If you are uncertain about your answers to these questions, you may return to page 1.4 and reconstruct this situation graphically.

Underestimates versus overestimates

Page 1.6 again displays the graph of the function $f(x) = x^3 - 3x^2 - 2x + 6$, but this time, we are looking at the linear approximation at $a = 1$.

- As point *p* is located to the left of the point of tangency, is the approximation an *overestimate* or an *underestimate*?

- Drag point *p* towards the point of tangency and observe the error associated with the linear approximation. What happens as point *p* moves to the *right* of the point of tangency?

- What is the significance of the point of tangency?

- Generalize your findings by explaining when a linear approximation produces an *overestimate* and when a linear approximation produces an *underestimate*.

Finding intervals of accuracy

Consider the following question:

How close to –1 must *x* be for the linear approximation of $f(x) = x^3 - 3x^2 - 2x + 6$ at $a = -1$ to be within 0.2 units of the *true* value of f(*x*)?

This question will be explored graphically using the diagram on page 1.7. The equations of the graphs shown are $f(x) = x^3 - 3x^2 - 2x + 6$, $g(x) = f(x) + 0.2$, and $h(x) = f(x) - 0.2$. The tangent to f(*x*) at $a = -1$ is also displayed.

- How can you use the transformed graphs g(*x*) and h(*x*) to answer the question posed in this problem?

- How close to –1 must *x* be for the linear approximation of $f(x) = x^3 - 3x^2 - 2x + 6$ at $a = -1$ to be within 0.2 units of the *true* value of f(*x*)?

There is a similar diagram on page 1.8, although this time, the point of tangency is at $a = 1$.

- Explain how this situation differs from the one posed on page 1.7.

- Use graphical methods to determine an interval that will ensure that the linear approximation at $a = 1$ is accurate to within 0.2 units of f(*x*).