NUMB3RS Activity: The Königsberg Bridge Problem
Episode: “Toxin”

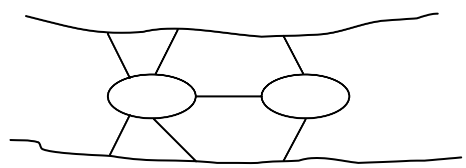
Topic: Graph Theory **Grade Level**: 6 - 12

Objective: Paths on Networks or Graphs

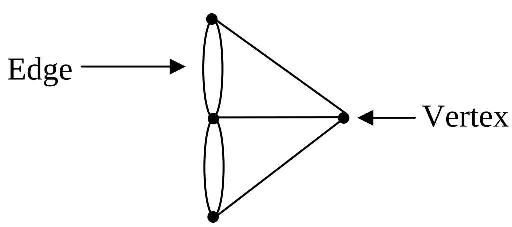
Time: 30 minutes

Introduction

Charlie discusses the “Seven Bridges of Königsberg,” a classic mathematics puzzle investigated by Leonhard Euler (1707–1783), as an inspiration for tracking a serial poisoner. The river Pregel divided the town of Königsberg, Germany into four separate landmasses connected by seven bridges as shown below.



As curious people, the citizens of Königsberg wondered if there was a path that would allow them to cross all seven bridges without crossing any one bridge twice. They were unable to find such a path. Euler was intrigued by the issue and in the process of working on it, he began a new field now called “network theory.” He generalized the map of Königsberg into the following diagram, where each landmass is a vertex and each bridge is an edge.



Euler’s work has been generalized in the following definitions:

1. Graph: a figure made up of vertices and edges.
2. Odd and even vertices: a vertex is odd if it has an odd number of edges emanating from it, otherwise it is even.
3. Euler path: a continuous path that passes through every edge of a graph once and only once.
4. Euler cycle (or circuit): a path through a graph which starts and ends at the same vertex and includes every edge exactly once.
5. Euler trail: a path that goes through each edge of a graph exactly once and such that the start and end vertices are different.
6. Network: a figure made up of points or vertices connected by non-intersecting curves or edges.

Euler proved the following theorems:

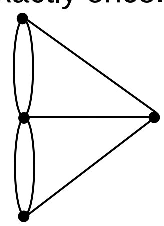
Theorem 1: If a graph has more than two odd vertices, then it does not have an Euler path.
Theorem 2: If a graph has two or fewer odd vertices, then it has at least one Euler path.

Students will investigate these theorems in the activities.

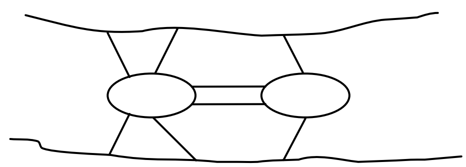
Discuss with Students

Questions about the Königsberg Bridge problem that could be considered follow:

1. Try to find a path that allows all landmasses to be traversed as often as needed and all bridges to be crossed exactly once.



2. If another bridge were to be added between the two islands (the ovals), could the desired walk be achieved?



3. Can a graph with exactly two odd vertices have an Euler path?

4. Can a graph with more than two odd vertices have an Euler path?

Discuss with students answers: 1. *There is no path that allows the walk as described.* **2.** *The walk can be accomplished*. **3.** *Yes* **4.** *No*

Student page answers: 1. a. *no path is possible; 4 odd vertices* **b.** *start at any vertex and move around the figure; 0 odd vertices* **c.** *start at any vertex and complete the inside star then move around the circle; 0 odd vertices* **d.** *no path is possible; 4 odd vertices* **e.** *start at one of the two odd innermost vertices and end at the other; 2 odd vertices* **f.** *start at one of the two odd vertices and end at the other; 2 odd vertices* ***2.*** **a.** *Euler trail - no; Euler cycle - no* **b.** *Euler trail - no; Euler cycle - yes* **c.** *Euler trail - yes; Euler cycle - no* **d.** *Euler trail - yes; Euler cycle - no* **e.** *Euler trail - no; Euler cycle - no* **3.** *Answers may vary. Students should add or delete bridges so that there are two or fewer odd vertices.*

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Name: ________________________________ Date: _______________

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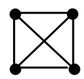
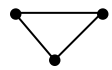
1. If possible, draw an Euler path that crosses each bridge (edge) to the islands (vertices) without lifting your pencil and without going over any bridge more than once. If you found a path, how many odd vertices are there?
(Note: If there is no black dot, the edges do not intersect.)

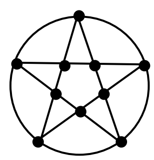
a.



b.



c.



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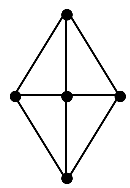
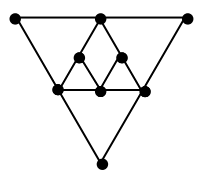
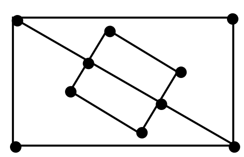


e.



f.



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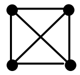
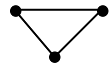
2. Determine if each one of the following is has an Euler trail, an Euler cycle, or neither.
(Note: If there is no black dot, the edges do not intersect.)

a.

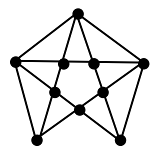


b.

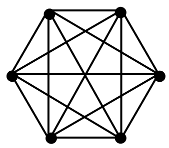


c.

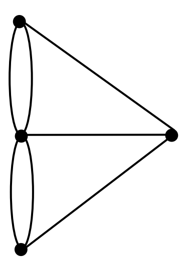
d.



e.



3. Experiment with creating and eliminating bridges from Euler's original problem. What is the simplest design you can find that would allow the residents to cross each bridge only once? Can you prove that your solution is valid?



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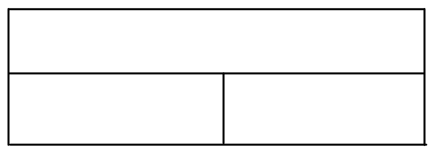
The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Extension 1
Euler's graph theory can be used on other traditional problems. For instance, the drawing can be considered to be composed of ten separate segments. Is possible to draw a continuous curve that passes through each of the line segments exactly once without passing through a vertex?



Use Euler's theorems to create your own graph challenge.
http://www.jcu.edu/math/vignettes/bridges.htm

Extension 2
When Leonhard Euler presented his solution of the Königsberg Bridge problem to the Russian Academy in 1735, it set other ideas into motion, initiating the mathematics of **topology**. Investigate Topology and its relationship to Euler's theorems.

References:
A History of Topology at
http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Topology_in_mathematics.html

Geometry and Topology
http://www.maths.warwick.ac.uk/gt

What is Topology?
http://neil-strickland.staff.shef.ac.uk/Wurble.html

Additional Reference:
Moscovick, Ivan, and David Brian. "Mindgames: Network Games." New York: Workman Publishing, 2001

For a related problem go to: **http://www.contracosta.cc.ca.us/math/konig.htm**

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