Math Objectives

- Students will produce various graphs of Taylor polynomials.
- Students will discover how the accuracy of a Taylor polynomial is associated with the degree of the Taylor polynomial.
- Students will visualize the accuracy and relate this to symmetry, arc length, and a point of discontinuity.
- Students will examine certain common functions and draw specific conclusions about the Taylor polynomial associated with these functions.
- Students will look for and make use of structure. (CCSS Mathematical Practice)
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)

Vocabulary

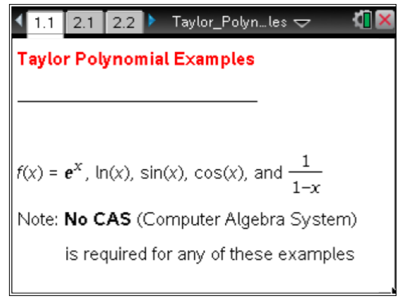
- Taylor polynomial
- degree *n*
- accuracy of the approximation
- neighborhood of *a*

About the Lesson

- This lesson involves Taylor polynomials associated with five common functions.
- As a result, students will:
  - Learn about Taylor polynomials graphically and numerically.
  - Conjecture about factors that affect the accuracy of Taylor polynomial approximations.
  - Visualize the effects of *n* and *a* on the accuracy of the approximation.

TI-Nspire™ Navigator™ System

- Use Screen Capture to demonstrate various approximations depending on the degree and *a* (page 2.3).
- Use Teacher Edition computer software to illustrate various Taylor polynomials.
- Encourage students to try several different values of *n* and *a.*



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Manipulate a slider

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- In *Graphs*, you can view the function entry line by pressing ctrl G, and then enter a function.
- Press ctrl doc▾ and select Lists & Spreadsheets to insert a new Lists & Spreadsheets page.

Lesson Materials:
Student Activity
Taylor_Polynomial_Examples.pdf
Taylor_Polynomial_Examples.doc
TI-Nspire document
Taylor_Polynomial_Examples.tns

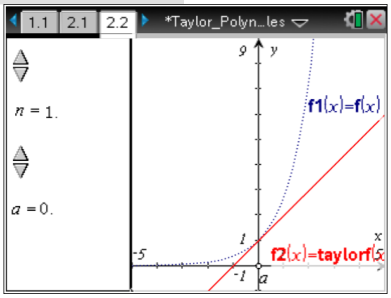
Visit www.mathnspired.com for lesson updates and tech tip videos.

- Use Screen Capture to display Taylor polynomials associated with $y = \sin x$ in order to discover the symmetry and the changes in the polynomials as *n* increases by 1.
- At the end of this activity, ask students to consider other common functions and the associated Taylor polynomials.

Discussion Points and Possible Answers

Tech Tip: If you use the slider to move away from *a* = 0 and then back to *a* = 0, occasionally the calculator will display a very small number, not exactly 0. There may be several reasons for this approximation that can lead to interesting discussions. Ask students to determine the increment in the examples involving $y = \sin x$ and $y = \cos x$, and why this increment was selected.

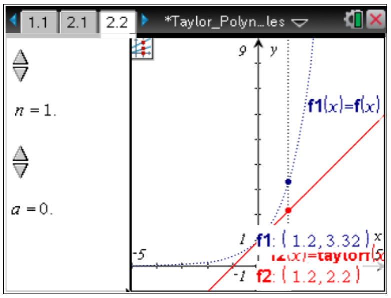
Move to page 2.2.



1. In the first example, the graph of $y = e^x$ is dotted and the graph of the Taylor polynomial of degree *n* at *a* is solid. Use the slider arrows to change the degree, *n*, or the value of *a*.

   a. With *a* = 0, set *n* = 1 Graph the first degree Taylor polynomial, T_1, at 0. Describe the graph of $y = T_1(x)$.

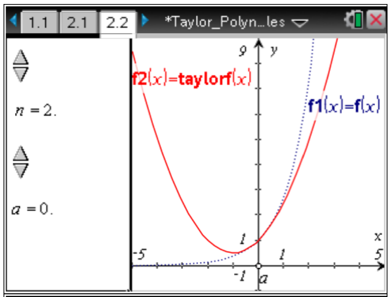
   Answer: The graph of the first degree Taylor polynomial is a straight line, the tangent line to the graph of $y = e^x$ at the point (0,1).



   b. Use the graph of $y = T_1(x)$ and the Trace All feature to describe the accuracy of the Taylor polynomial approximation as *x* moves farther from *a* = 0.

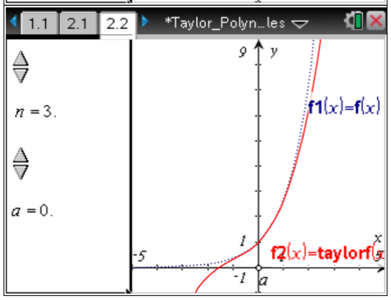
   Answer: As *x* moves farther from *a* = 0, the approximation becomes less accurate. This is confirmed by visual inspection and numerically using the Trace All feature.

c. Set *n* = 2. Describe the graph of $y = T_2(x)$, the second degree polynomial at 0.



Answer: The graph of the second degree Taylor polynomial is a parabola.

d. Set *n* = 3. Describe the graph of $y = T_3(x)$, the third degree polynomial at 0.



Answer: The graph of the third degree Taylor polynomial looks like the graph of $y = x^3$, a cubic polynomial, shifted vertically up 1 unit.

e. Consider the graph of other Taylor polynomials for $n \geq 4$. Describe the accuracy of the Taylor polynomial approximation as *n* increases.

Answer: As *n* increases, the Taylor polynomial approximation becomes more accurate. For larger values of *n*, the graph of the Taylor polynomial seems to lie more closely on the graph of $y = e^x$, especially to the left of *a* = 0.

Move to page 2.3.

On this *Lists & Spreadsheets* page you may enter values for x in column A. The following values will be computed automatically: $\mathbf{f}(x)$, $\mathbf{taylorf}(x)$, and $|\,\mathbf{f}(x) - \mathbf{taylorf}(x)\,|$, columns B, C, and D respectively. These resulting values are dependent upon the current values of *n* and *a*.

2. Adjust the values of *n* and *a* on page 2.2 as necessary and use the *Lists & Spreadsheets* page to answer the following questions.

a. For a fixed value of *n*, describe the accuracy of the Taylor polynomial approximation as the values of *x* are farther away from *a*.

Answer: As *x* moves farther away from *a*, the approximation is worse. The closer *x* is to *a*, the more accurate the approximation given by the Taylor polynomial.

b. For fixed values of *a* and *x*, describe the accuracy of the Taylor polynomial approximation as *n* increases.

Answer: For fixed values of *a* and *x*, as the degree of the Taylor polynomial increases, that is, as *n* increases, the approximation given by the Taylor polynomial becomes more accurate.

Move to page 3.2.

4. In this example, the graph of $y = \ln(x)$ is dotted and the graph of the Taylor polynomial of degree *n* at *a* is solid. Use the slider arrows to change the degree, *n*, or the value of *a*. Adjust the values of *n* and *a* as necessary to answer the following questions.

   a. For *a* = 2, describe the accuracy of the Taylor polynomial approximation as *n* increases.
   Answer: As *n* increases, the Taylor approximation becomes more accurate. As *n* increases, to the left of *a* = 2, the graph of the Taylor polynomial lies closer to the graph of $y = \ln x$ and always decreases without bound as $x \to 0^+$. As *n* increases, to the right of *a* = 2, the graph of the Taylor polynomial appears to be a very good approximation up to approximately *x* = 4.

   b. Describe the behavior of each Taylor polynomial as $x \to -\infty$ and as $x \to +\infty$. What happens to the graph of the Taylor polynomial, as $x \to +\infty$, as *n* increases by 1, for example, from *n* = 6 to *n* = 7? Explain why this behavior alternates as *n* increases.
   Answer: The right tail of the graph of the Taylor polynomial alternates: for *n* even, as $x \to +\infty$, *taylorf* $\to +\infty$, and for *n* odd, as $x \to -\infty$, *taylorf* $\to -\infty$. As *n* increases by 1, an additional term is added to the Taylor polynomial, of the form $\frac{f^{(n)}(2)}{n!}(x-2)^n$. This term dominates, or controls, the behavior of the polynomial for large values of *x*. Therefore, as *x* increases, $(x-2)^n$ becomes large positive, and the behavior of the Taylor polynomial is controlled by the sign of the *n*th derivative at 2. This derivative term alternates in sign, causing the graph of the Taylor polynomial to alternate as *x* increases.

   c. For *a* = 0.3, consider various Taylor polynomials of different degrees. Explain why the Taylor polynomial appears to be a very good approximation to the left of *a* = 0.3 but diverges rapidly to the right of *a* = 0.3.
   Answer: Although it will become clearer in the next example, the Taylor approximation is accurate on a symmetric interval about the point *a*. In this example, for *a* = 0.3, the Taylor polynomial can only provide an accurate approximation up to (but not including) *x* = 0.6. This is because the function $\mathbf{f}(x) = \ln x$ has domain $x > 0$.

> **Teacher Tip:** Ask students to adjust the Window Settings in order to more closely observe the changes to the graph of the Taylor polynomial as *n* increases. The approximation to the left of *a* = 0.3 might appear to be *better*. The approximation is accurate on a symmetric interval about *a* = 0.3. However, the arc length of the graph of $y = \ln x$ to the left of *a* = 0.3 is greater than for a similar interval to the right of *a* = 0.3. This leads to the graphical suggestion that the approximation is better to the left of *a* = 0.3.

Move to page 4.2.

5. In this example, the graph of $y = \sin(x)$ is dotted and the graph of the Taylor polynomial of degree *n* at *a* is solid. Use the slider arrows to change the degree, *n*, or the value of *a*.

   a. For *a* = 0 and *n* = 1, describe the graph of the Taylor polynomial. Find the Taylor polynomial and describe the approximation for sin *x* for *x* close to 0.
   Answer: The graph of the Taylor polynomial is a straight line through the origin with slope 1. The Taylor polynomial is $\textbf{taylorf}(x) = x$. For *x* close to 0, this suggests $\sin x \approx x$.

   b. For *a* = 0, consider the graph of the Taylor polynomials as *n* increases. Explain why the graph of the Taylor polynomials for *n* = 1 and for *n* = 2 are identical, and for *n* = 3 and *n* = 4, etc.
   Answer: For *i* even, $\mathbf{f}^{(i)}(x) = \pm\sin x$ and $\mathbf{f}^{(i)}(0) = 0$. Therefore, there are no terms of even degree in any Taylor polynomial for $y = \sin x$. The Taylor polynomials for *n* = 3 and *n* = 4, for example, are identical.

   c. For each value of *a* and *n*, describe the accuracy of the Taylor approximation about the point *x* = *a*.
   Answer: The Taylor polynomial appears to be accurate on a symmetric interval about the point *x* = *a*.

Move to page 5.2.

6. In this example, the graph of $y = \cos x$ is dotted and the graph of the Taylor polynomial of degree *n* at *a* is solid. Use the slider arrows to change the degree, *n*, or the value of *a*.

   a. For *a* = 0 and *n* = 1, describe the graph of the Taylor polynomial. Find the Taylor polynomial and explain why the slope of this linear approximation is 0.
   Answer: The graph of the Taylor polynomial is a horizontal line through the point (0, 1). The Taylor polynomial is $\textbf{taylorf}(x) = 1$. $\mathbf{f}'(x) = -\sin x$ and $\mathbf{f}'(0) = 0$. Therefore, the coefficient on the linear term is 0; the slope of this linear approximation is 0.

   b. For *a* = 0, consider the graph of the Taylor polynomials as *n* increases. Explain why the graph of the Taylor polynomials for *n* = 0 and for *n* = 1 are identical, and for *n* = 2 and *n* = 3, etc.

   Answer: For *i* odd, $\mathbf{f}^{(i)}(x) = \pm\sin x$ and $\mathbf{f}^{(i)}(0) = 0$. Therefore, there are no terms of odd degree in any Taylor polynomial for $y = \cos x$. The Taylor polynomials for *n* = 4 and *n* = 5, for example, are identical.

Move to page 6.2.

7. In this example, the graph of $y = \frac{1}{1-x}$ is dotted and the graph of the Taylor polynomial of degree *n* at *a* is solid. Use the slider arrows to change the degree, *n*, or the value of *a*.

   a. For *a* = 0, consider various Taylor polynomials of degree *n*. Explain why there is no graph of the Taylor polynomial to the right of *x* = 1.
   <u>Answer:</u> The function $\mathbf{f}(x) = \frac{1}{1-x}$ has a discontinuity at *x* = 1. The graph of the Taylor polynomial cannot extend beyond this discontinuity.

   b. Consider the graph of the Taylor polynomial for *a* = 0 and *n* = 7. Explain the accuracy of this Taylor polynomial. Why does the Taylor polynomial appear to be a much better approximation to the right of *a* = 0than to the left?
   <u>Answer:</u> The arc length of the graph of $y = \frac{1}{1-x}$ is greater for $0 \le x < 1$ than for $-1 < x \le 0$.
   Therefore, the Taylor polynomial appears to be a better approximation to the right of *a* = 0. The approximation is accurate on a symmetric interval about *a* = 0.

   c. Explain how to obtain the graph of a Taylor polynomial that can be used to approximate the portion of the graph of $y = f1(x)$ to the right of *x* = 1.
   <u>Answer:</u> Select a value *a* > 1. This will produce a Taylor polynomial that can be used to approximate the portion of the graph of $y = f1(x)$ to the right of *x* = 1.

Wrap Up

Upon completion of this activity, the teacher should ensure that students understand:

- How the accuracy of a Taylor polynomial is associated with the degree of the Taylor polynomial and the value *a*.
- A Taylor polynomial is accurate on a symmetric interval about *x* = *a*.
- How a point of discontinuity affects a Taylor polynomial.
- The relationship between the *i*th derivative of the function and the *i*th derivative of the corresponding Taylor polynomial.

At the end of this activity, you might consider asking students to graph other common functions and the associated Taylor polynomials. For example, consider $\mathbf{f}(x) = \tan^{-1} x$, $y = e^{-x^2}$, and $y = e^x \sin x$.