Cubic Regions

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Abstract: This activity is an applications of derivatives and of definite integrals. It introduces students to an interesting property of cubics and a method of proving that property using the TI-89 scripts. They then use the symbolic capacity of their calculator to generalize upon specific results.

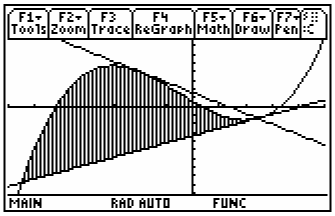
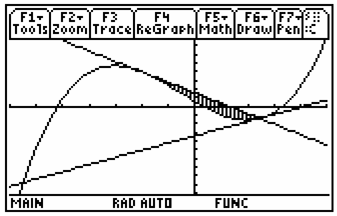
NCTM Principles and Standards:
Algebra standards
a) Understand patterns, relations, and functions
b) generalize patterns using explicitly defined and recursively defined functions;
c) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
d) use symbolic algebra to represent and explain mathematical relationships;
e) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
f) draw reasonable conclusions about a situation being modeled.

Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard
a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

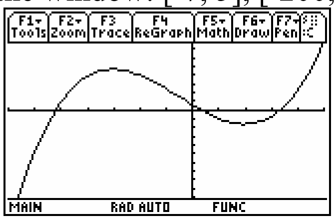
Key topic: Applications of derivatives. Tangent line. Application of Definite Integrals - area between curves. Scripts, formal proofs.

Degree of Difficulty: Advanced
Needed Materials: TI-89 calculator
Situation: Cubic polynomials have many interesting properties. In this activity we'll use calculus to investigate one of them with the aid of the TI-89 calculator. Consider a line tangent to a cubic. Where else does this line cross the cubic? Construct a line tangent to the cubic at that point, too. Two regions are formed. What is the ratio of their areas as indicated by the two shaded regions in these graphs?

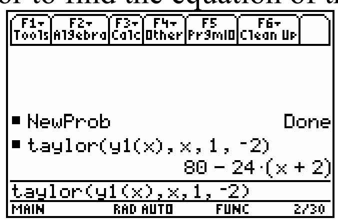



Step-By-Step Solution with a particular example:

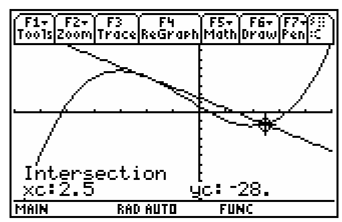
Take a look at the graph of a particular cubic polynomial: $y = 2x^3 + 3x^2 - 36x + 12$. Using the window: [-7, 5], [-200, 150]



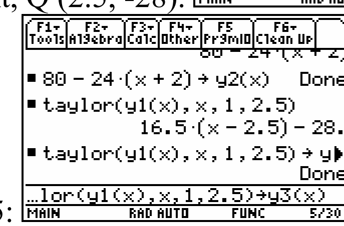
Pick a point P on the graph, say (-2, 80) and draw the tangent line, L, through P to the graph. You can get the calculator to find the equation of the tangent line by using a Taylor polynomial of degree 1:

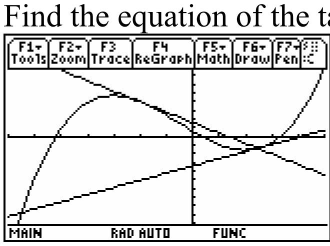


Now L will intersect the graph at another point, Q (2.5, -28).

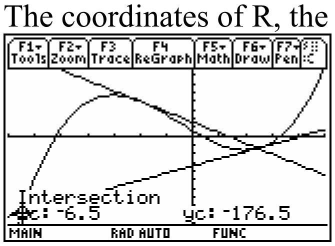


Find the equation of the tangent line, M, at 2.5:



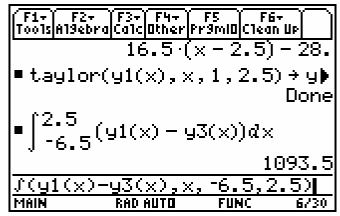


The coordinates of R, the other point where M crosses the curve, are (-6.5, -176.5)

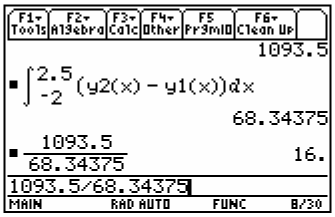


The area between line M and the curve C is

$$\int_{-6.5}^{2.5}((2x3+3x2-36x+12)-(16.5(x-2.5)-28))dx = 1093.5$$

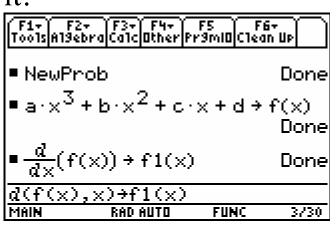


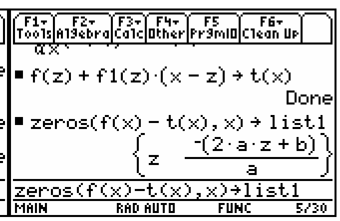
The area between line L and the curve is $\int_{-2}^{2.5}((80-24(x+2)-(2x3+3x2-36x+12))dx =$ 68.34375

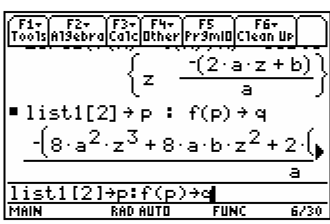


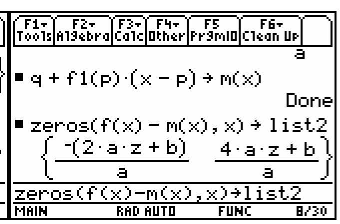
Compute the ratio of the two areas: $\frac{1093.5}{68.34375} = 16$. Does this always work?

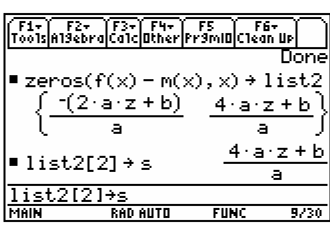
Let's look at this more analytically. Here we are writing essentially the same commands as we did with the numerical example - but with variables. In the process, we are creating a proof of the fact that the ratio of the areas of these two regions is always 16:1 no matter what the equation of the cubic is and no matter what point (z, f(z)) we pick on it!

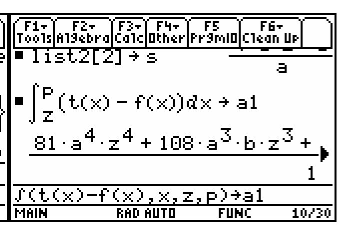


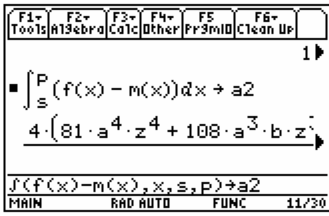








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∫(f(x)-m(x),x,s,p)→a2
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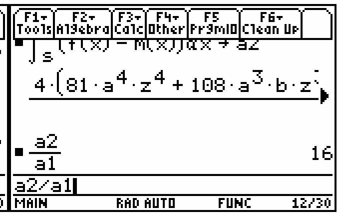
F1 Tools F2 Algebra F3 Calc F4 Other F5 PrgmIO F6 Clean Up
16
a2/a1
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