Fundamental Algebra

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Answers

7 8 9 10 11 **12**

TI-Nspire | Investigation | Student | 45 min

Introduction

The quadratic equation $f(x)=x^2+8x+12$ has two distinct linear factors: $x+2$ and $x+6$ which means it has two distinct roots: $x=-2$ and $x=-6$. The quadratic equation $g(x)=x^2+8x+16$ has one 'repeated' linear factor: $x+4$ and therefore one distinct (but repeated) root: $x=-4$. The quadratic equation: $h(x)=x^2+8x+17$ has no real distinct factors or roots but does have two distinct complex factors: $x+4-i$ and $x+4+i$, and therefore two distinct complex roots $x=-4+i$ and $x=-4-i$.

This investigation looks at factors and roots of larger polynomials with **real** coefficients.

Real Preliminary Questions

Question: 1

a. Where possible, factorise each of the following quadratics over R.

$f(x)=x^2+8x+12$
$=(x+6)(x+2)$

$g(x)=x^2+8x+16$
$=(x+4)^2$

$h(x)=x^2+8x+17$
$=\textit{No real factors}$

$a(x)=x^2-4x-5$
$=(x-5)(x+1)$

$b(x)=x^2-6x+9$
$=(x-3)^2$

$c(x)=x^2-10x+29$
$=\textit{No real factors}$

b. Each of the quadratic functions above is used to produce a polynomial (below). Complete the table with factors and roots determined over the real number system.

Equation:	Degree:	Factors:	Distinct Roots:
$p(x)=f(x)\cdot a(x)$	4	$(x+6)(x+2)(x+1)(x-5)$	$x=-6,-2,-1,5$
$q(x)=f(x)\cdot g(x)$	4	$(x+6)(x+2)(x+4)^2$	$x=-6,-2,-4$
$r(x)=f(x)\cdot h(x)$	4	$(x+6)(x+2)(x^2+8x+17)$	$x=-6,-2$
$s(x)=g(x)\cdot b(x)$	4	$(x+4)^2(x-3)^2$	$x=-4,3$
$t(x)=h(x)\cdot c(x)$	4	$(x^2-10x+27)(x^2+18x+17)$	$x=\textit{No real roots}$

c. The functions $r(x)$ and $s(x)$ have the same number of distinct roots yet they are significantly different. Explain the difference with reference to the number and type of roots.
$r(x)$ has two distinct roots, the function crosses the x axis at these points. $s(x)$ has two repeated roots, the function is tangent to the x axis at these points. (Does not cross).

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Complex Preliminary Questions

Question: 2

a. Factorise each of the following quadratics over C.

$f(z)=z^2+8z+12$
$=(z+6)(z+2)$

$g(z)=z^2+8z+16$
$=(z+4)^2$

$h(z)=z^2+8z+17$
$=(z+4-i)(z+4+i)$

$a(z)=z^2-4z-5$
$=(z-5)(z+1)$

$b(z)=z^2-6z+9$
$=(z-3)^2$

$c(z)=z^2-10z+29$
$=(z-5+2i)(z-5-2i)$

b. Each of the quadratic functions above is used to produce a polynomial (below). Complete the table with factors and roots determined over the complex number system.

Equation:	**Degree:**	**Factors:**	**Distinct Roots:**
$p(z)=f(z)\cdot a(z)$	4	$(z+6)(z+2)(z+1)(z-5)$	$z=-6,-2,-1,5$
$q(z)=f(z)\cdot g(z)$	4	$(z+6)(z+2)(z+4)^2$	$z=-6,-2,-4$
$r(z)=f(z)\cdot h(z)$	4	$(z+6)(z+2)(z+4-i)(z+4+i)$	$z=-6,-2,-4\mp i$
$s(z)=g(z)\cdot b(z)$	4	$(z+4)^2(z-3)^2$	$z=-4,3$
$t(z)=h(z)\cdot c(z)$	4	$(z^2-5+2i)(z-5-2i)(z+4-i)(z+4+i)$	$z=-5\mp 2i,-4\mp i$

c. Comment on the number of factors for each of the polynomials.
Each polynomial has 4 factors if ‘repeated’ factors are counted as separate factors. Complex factors occur in conjugate pairs.

Multiplicity

The polynomial $p(z)=z^2-6z+9$ has one distinct root: $z=3$. The root $z=3$ has multiplicity 2 since it occurs twice. The polynomial $p(z)$ is said to have 2 roots when degenerate roots are included, that is the repeated roots are counted with their multiplicity.

d. Comment on the number of roots for each of the polynomials including degenerate roots.
Each polynomial has 4 roots if ‘repeated’ roots are counted with their multiplicity. Complex roots occur in conjugate pairs.

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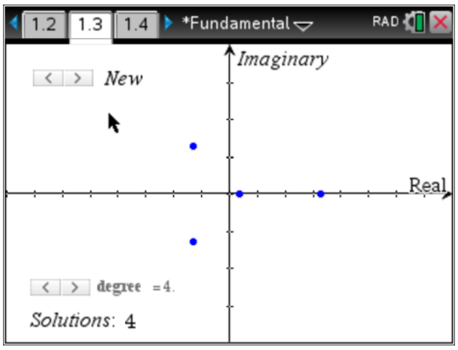
Investigation

Open the TI-nspire document: “Fundamental” and navigate to page 1.3. The ‘degree’ slider produces a random polynomial of the nominated degree with **real coefficients**. The ‘new’ slider can be used to generate another polynomial of the same degree. The polynomial is factorised and the roots plotted on the Argand plane. The example shown opposite is for a polynomial of degree 4 that has two real roots and two imaginary roots.



Question: 3

Generate at least 10 more polynomials of degree 4 and describe the number and type of roots for each polynomial.

There are always 4 roots. The roots occur in three different patterns:

- All roots are real
- Two real roots and one pair of complex conjugate roots
- All roots are complex and occur in conjugate pairs

Occasionally only three roots appear, but the missing root must be a repeated real root as the complex roots are shown in their conjugate pair.

Question: 4

Generate at least 10 polynomials of degree 5 and describe the number and type of roots for each polynomial. Comment on the number of real and complex roots.

There are always 5 roots. The roots occur in three different patterns:

- All roots are real
- Three real roots and two complex (conjugate roots)
- One root is real and two pairs of complex conjugate roots

Note: It is very rare in the randomly generated roots for repeated real roots to occur, occasionally one of the real roots may be too large to display on the axis. Students should identify that the missing root is real and how they know that it is real.

Question: 5

Generate at least 10 more polynomials of degree 6 and describe the number and type of roots for each polynomial.

There are always 6 roots. The roots occur in four patterns:

- All roots are real
- Four real roots and one pair of complex conjugate roots
- Two real roots and two pairs of complex conjugate roots
- Three pairs of complex conjugate roots

The random generation of polynomials rarely produces all real roots; however students should be able to reason this from their own experience and knowledge.

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Question: 6

Generate polynomials of various degrees and identify any patterns with respect to the following:

- Type and number of roots (Real / Complex)
- If roots are counted with their multiplicity the number of roots is equal to the degree of the polynomial.
- Nature of complex roots
- Complex roots always occur in conjugate pairs as their product must remove imaginary components if the coefficients of the polynomial are all to be real.

Question: 7

A polynomial of degree 6 with real coefficients has the following roots: $z = 3 + i$, $z = 2 - i$ and $z = 6$. Which one of the following could represent the remaining roots:

a) $z = 3 + i$, $z = 2 - i$ and $z = 4$

b) $z = 3 - i$, $z = 2 + i$ and $z = 4$

c) $z = -3 + i$, $z = -2 - i$ and $z = 4$

d) $z = 3 + i$, $z = 2 - i$ and $z = -4$

e) $z = 3 - i$, $z = 2 + i$ and $z = 4 - i$

Correct Answer is B. Complex roots are conjugates to those provided and the remaining root is real.

Question: 8

The polynomial $p(z) = z^6 + az^5 + bz^4 + cz^3 - 18z^2 - dz - 1500$ has roots: $z = -3 + i$, $z = -4 - 3i$ and $z = 2$. Determine the values of *a*, *b*, *c* and *d*.

Since all the coefficients of *p(z)* are real, expand the factorised polynomial (using the roots provided) including conjugate roots: $(z + 3 - i)(z + 3 + i)(z + 4 - 3i)(z + 4 + 3i)(z - 2)(z - n)$

By either equating coefficients or using: *p*(0) = -1500 results with *n* = 3. Substitute n = 3 back into the expression and expand again: $(z + 3 - i)(z + 3 + i)(z + 4 - 3i)(z + 4 + 3i)(z - 2)(z - n) \mid n = 3$

$a = 15,\ b = 91,\ c = 229 \text{ and } d = -1130$, note that the z^2 coefficient helps validate solution.

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The QR Code provided below is a link to a wonderful video on the Fundamental Theorem of Algebra. The video is produced by a team of mathematicians that have many equally brilliant videos on YouTube under the name “Numberphile”. The explanation is very clear and provided at a level suitable for students and teachers. I particularly like the use of two Argand planes to illustrate the mapping of functions.

https://www.youtube.com/watch?v=shEk8sz1oOw

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