NUMB3RS Activity: When Does 1 + 1 ≠ 2?
Episode: "Convergence"

Topic: A Little Group Theory **Grade Level**: 8 - 12
Objective: To consider contexts in which 1 + 1 $\neq$ 2.
Time: About 20 minutes
Materials: graphing calculator

Introduction

In "Convergence", Charlie writes "1 + 1 = 2" on the board and gives his students the homework assignment of explaining why he is right and why he is wrong. Most students are familiar with the addition of real numbers. Some might even be familiar with addition of complex numbers. Each of these sets of numbers forms a *group* under the binary operation of addition, because addition satisfies four basic axioms when applied to the set:

- **Closure:** If a and b are in the set, then $a + b$ is in the set;
- **Associativity:** For any a, b, and c in the set, $a + (b + c) = (a + b) + c$;
- **Identity:** There is an additive identity (which we can call 0) such that $a + 0 = 0 + a = a$ for all a;
- **Inverses:** Every element a has an additive inverse (which we can call $-a$) such that $a + (-a) = (-a) + a = 0$.

It is true in each of the familiar number systems mentioned above that 1 + 1 = 2, so it is fairly easy to see why Charlie is "right." (*Proving* he is right is another matter. Bertrand Russell and Alfred North Whitehead used principles of logic to justify arithmetic in their classic 3-volume work *Principia Mathematica* (1910-1913), but it took them 362 pages to arrive at the result "1 + 1 = 2"!) Nevertheless, there are useful number systems in which 1 + 1 $\neq$ 2, most notably the 2-element group.
What follows are real-world contexts in which we *interpret* 1 + 1 as a number different from 2. Some examples are explored, but you and your students are encouraged to find more. In the Extensions, students learn enough about group theory to understand that there is some logic in defining 1 + 1 to be 0 if a group has only two elements, 0 and 1.

Discuss with Students

When you calculate an answer for a math question, is it the final answer? Not always! For example, if you are finding the average height of a group of adults you may get 69.857 inches, but you would probably say that 70 inches is the average height. The answers are dependent on the assumptions made on the data and what makes sense for the problem posed. In this activity, you will discover that the most basic of all questions 1+1, may not always equal 2 depending on your context.

Student page answers: **1.** *1 dozen + 1 dozen = 24* **2.** *If x is the cost of 1 CD, then 1x + 1x = 1x, since the second CD is free.* **3**. *The man lives at 11 Oak Street and is buying numerals to stick onto his mailbox* **4**. *Sam will have walked 1 + 1 = 2 miles, but the distance between the two houses, by the Pythagorean Theorem, is* $\sqrt{2}$ *miles (≈ 1.414).* **5.** *60° (Sam's walk and the distance between the houses determine an equilateral triangle 1 mile on each side).*
6. *{1.4 1.3 2.7}; {1 1 3}; 1.4 and 1.3 both round to 1, but the sum 2.7 rounds to 3.* **7**. *Answers will vary. One possibility is A = 0.7 and B = 0.6.* **8**. *Any number that rounds to 1 must be less than 1.5, so the sum of two such numbers will be less than 3 and will not round to 4.*

Dan Kennedy, Baylor School, Chattanooga, TN

Name: ___________________________________ Date: ________________

NUMB3RS Activity: When Does 1 + 1 ≠ 2?

Charlie writes "1 + 1 = 2" on the board and asks his students to explain why he is right—and why he is wrong. You use the idea that 1 + 1 = 2 every day—let’s explore some situations when 1 + 1 ≠ 2.

Playing with Units

1. The Smiths run out of eggs at breakfast. Mr. Smith buys one dozen eggs on his way home from work, unaware that his wife is also buying one dozen eggs on *her* way home from work. Explain in this context why 1 + 1 = 24.

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2. Seedy CD's is having a sale where you can buy any one of the top ten sellers and get a second CD for free. Explain in this context why 1 + 1 = 1.

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3. A man goes into a hardware store, holds up a common item, and asks, "How much does 1 cost?" The clerk replies, "Fifty cents." The man says, "Fine. I’ll take an 11." The clerk replies, "That will be one dollar." This makes perfect sense, because in this context 1 + 1 = 11. What is this man buying?

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Playing with Geometry

4. To get from his house to Takisha's, Antonio walks 1 mile west on Maple Street, then 1 mile north on Atlantic Avenue. What is the distance between Antonio's house and Takisha's? Explain in this context why 1 + 1 = $\sqrt{2}$.

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5. If Maple Street were to intersect Atlantic Avenue at a certain non-right angle, Antonio's walk could show that 1 + 1 = 1. What would that angle be?

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Playing with Rounding

6. In many applications, numbers are rounded. For example, a $16.99 CD with a sales tax of 7.75% would cost $18.306725, but you will be charged $18.31 because costs are rounded to two decimal places.

On your calculator, let **A** = 1.4 and **B** = 1.3 by typing **1.4**→**A : 1.3**→**B** and pressing **ENTER**.

Then type **{A, B, A + B}** to see a list of A, B, and A + B.
Write the result here: ___________________________

Now go to the **MODE** menu and change the decimal display from **FLOAT** to **0**. (This will round everything to the nearest whole number.)

Type **{A, B, A + B}** and press **ENTER** again.
Write the result here: ___________________________

Explain in this context why 1 + 1 = 3.

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7. Find numbers A and B for which the same procedure suggests that 1 + 1 = 1.

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8. Explain why you cannot find numbers A and B for which the same procedure would suggest that 1 + 1 = 4.

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You should now change your decimal display mode back to **FLOAT**.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

- Playing in a basketball game, Tracy fouls Sam away from the basket. The referee awards Sam a "1 and 1." Explain in this context how Sam will ultimately determine whether 1 + 1 = 2, 1 + 1 = 1, or 1 + 1 = 0.

- A *group* is an algebraic system with one operation (like addition) that satisfies the four basic properties listed on the teacher page. The integers form a group under addition, and in that group 1 + 1 = 2. There is also an interesting group with only two numbers in it, in which addition is defined as in the table at right. Explain why the closure property *requires* that 1 + 1 $\neq$ 2. Explain why the inverse property *requires* that 1 + 1 = 0.

+	**0**	**1**
0	0	1
1	1	0

- Groups can also be defined with multiplication. The table at right shows the rules of multiplication on the set {1, –1}. Why could this set be considered algebraically equivalent to the set {0, 1} you saw above?

●	**1**	**–1**
1	1	–1
–1	–1	1

- As there is only one group with two elements, there is only one group with three elements. See if you can define the "rules of addition" for the group shown at right, using 0 as the additive identity.

+	**0**	**1**	**2**
0			
1			
2			

Related Topic

There are two different groups with four elements. One of them continues the pattern of the 2- and 3-element groups you have seen (called *cyclic* groups). The other is the *Klein Four-group*. All of the groups above satisfy the *Commutative* Property: $a + b = b + a$ for all elements a and b. Real number addition and multiplication are both commutative. Commutative groups are called *Abelian*, but not all groups are Abelian. The most familiar non-Abelian group is the multiplicative group of 2×2 matrices. The smallest *finite* non-Abelian group is the symmetric group S_3, which has 6 elements.

Additional Resources

A nice summary of the small finite groups considered in this activity can be found at **http://aleph0.clarku.edu/~djoyce/modalg/smallgroups.html**

Multiplication tables for all groups of orders 2 through 10 can be found at **http://www.math.niu.edu/~beachy/aaol/grouptables1.html**

For more on groups, see **http://en.wikipedia.org/wiki/Group_%28mathematics%29**

Dan Kennedy, Baylor School, Chattanooga, TN