NUMB3RS Activity: Let's Add Some Trig
Episode: "Counterfeit Reality"

Topic: Operations with trigonometric functions **Grade Level:** 11 - 12
Objective: Graph sums and products of trigonometric functions
Time: 15 - 20 minutes
Materials: TI-83 Plus/TI-84 Plus graphing calculator

Introduction

When Charlie explains how to crack a counterfeiting ring, he talks about the intricate engraving used in banknotes, called guilloche (pronounced "gee-YŌSH" – hard "g"). He explains that this is "created by the addition and multiplication of nested sine waves." In this activity students see what it means to combine trigonometric functions using addition and multiplication.

Discuss with Students

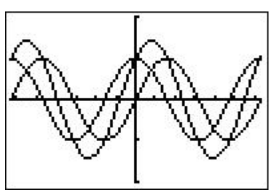
The concept of *function* is key in mathematics. Review "*f*(*x*)" notation and its meaning. Remind students that graphs of functions are the ordered pairs (x, $f(x)$), where $f(x)$ is the value of the function "*f*" for a given value of *x*. Review the basics of graphing sine curves if necessary. Emphasize that sin(x) is just one particular function, namely $f(x) = \sin(x)$, in the family of what are generally called "sine curves." To enhance the value of this activity, consider using the "Changing Sines" activity (also associated with this episode) as an introduction to graphing trigonometric functions. Note also that in this activity, angles are measured in radians.

Student Page Answers:

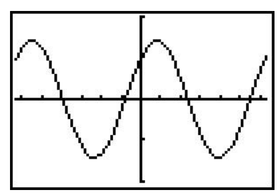
1. $f(2) = 2^2 = 4$, $g(2) = 2^3 = 8$, $h(2) = 2^2 + 2^3 = 4 + 8 = 12$, $k(2) = 2^2(2^3) = 4(8) = 32$.

2. $h\left(\frac{\pi}{6}\right) = sin\left(\frac{\pi}{6}\right) + cos\left(\frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} \approx 1.366$, $k\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} = \frac{1}{2}$.

3. *Screen looks like this:*

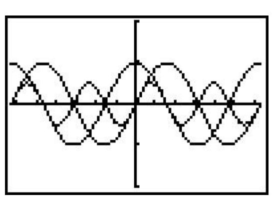


Final answer:



4a. *For that domain element, the product is zero since zero times any number equals zero. This occurs when the graph of either factor crosses the x-axis.* **4b.** *When one factor (function) has a value of 1 for a given domain element, the product is the value of the other factor (function) at that element.*

5. *Screen looks like this:*



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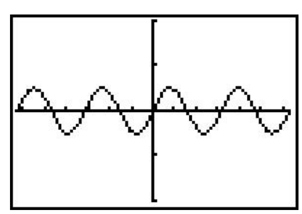

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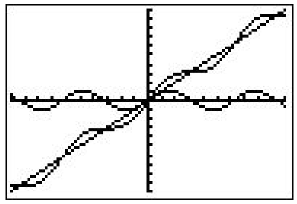
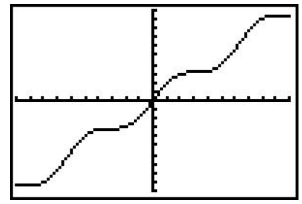
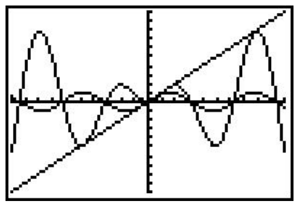
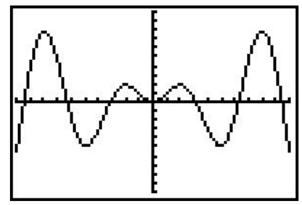
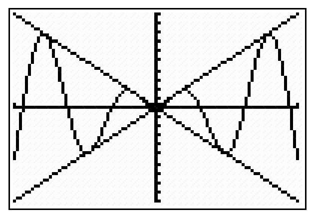
6. *Screen looks like this:*  *Final answer:* 

Whenever sin(x) = 0, the graph of the sum coincides with the graph of y = x.

7. *Screen looks like this:*  *Final answer:* 

8. *The graph remains between the two graphs of y = x and y = –x. Answers to "why?" will vary, but generally, since –1 ≤ sin(x) ≤ 1 (for all real values of x), the graph cannot get "outside" the boundary formed by these graphs, as shown:*



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Name: ______________________________ Date: _______________

NUMB3RS Activity: Let's Add Some Trig

When Charlie explains how to crack a counterfeiting ring, he talks about the intricate engraving used in banknotes, called guilloche (pronounced "gee-YŌSH" – hard "g"). He explains that this is "created by the addition and multiplication of nested sine waves." You know how to add and multiply numbers, but how is it done with functions?

Remember that a function is a set of ordered pairs, no two of which have the same first element. The first element is taken from a set called the "domain" and the second from the "range." With respect to the graph of a function, this means that no vertical line crosses the graph in more than one place. So for a given x, there is only one y [called $f(x)$] for which the ordered pair is on the graph. Adding two functions means to take every x value in the domain of the two functions, and add their corresponding y values to make the y value for the new function. With functions $f(x)$ and $g(x)$ where $h(x) = f(x) + g(x)$, for each value of x [for example, if $x = 5$, $h(5) = f(5) + g(5)$]. Multiplication works in the same way with multiplication replacing addition.

1. Suppose $f(x) = x^2$ and $g(x) = x^3$. If $h(x) = f(x) + g(x)$ and $k(x) = f(x)g(x)$, find $f(2)$, $g(2)$, $h(2)$, and $k(2)$.

2. The same thing can apply to trigonometric functions. Suppose $f(x) = \sin(x)$ and $g(x) = \cos(x)$. If $h(x) = f(x) + g(x)$ and $k(x) = f(x)g(x)$, find $h\left(\frac{\pi}{6}\right)$ and $k\left(\frac{\pi}{4}\right)$.

On graph paper, sketch what you think the graph of the function $y = h(x)$ in question 2 looks like.

3. To check your prediction with your calculator, enter $Y_1 = \sin(X)$, $Y_2 = \cos(X)$, and $Y_3 = Y_1 + Y_2$ (to enter Y_1 and Y_2, press [VARS], go to the **Y-VARS** menu and select **1:Function**). Press [MODE] and select **SEQUENTIAL**, and press [WINDOW] and change the settings to Xmin = -2π, Xmax = 2π, Ymin = -2, and Ymax = 2. Press [GRAPH] to see the functions graphed in order. Turn off Y_1 and Y_2 to see the final result.

4. **a.** What happens if one of the factors of the product of any two functions is 0? Where will this happen?
b. What if one of the factors is 1? Where will this happen?

5. Based on question 4, try to sketch the graph of $y = \sin(x)\cos(x)$. On your calculator, change equation Y_3 to $Y_3 = Y_1 * Y_2$. Graph all three equations to check your predictions, and then turn off Y_1 and Y_2 to see the final result.

6. Try to sketch the graph of $y = \sin(x) + x$. Again, think about what happens at the points suggested above. On your calculator, press [WINDOW] and change the settings to Xmin = -10, Xmax = 10, Ymin = -10, and Ymax = 10. To check your conjecture, leave Y_1 the same as in question 5, but change Y_2 to $Y_2 = X$ and Y_3 to $Y_3 = Y_1 + Y_2$.

7. Try to sketch the graph of $y = x\sin(x)$. On your calculator, change Y_3 to $Y_3 = Y_1 * Y_2$ and press [GRAPH] to check your prediction.

8. Leave all settings the same as in question 7, and enter $Y_4 = -X$. Draw the graphs of Y_2, Y_3, and Y_4. Describe what you notice about the relationship among these graphs.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

This activity is just an introduction to what it means to add and multiply functions (and in particular the sine waves Charlie explains). Combine these techniques with the changes of amplitude, period, and phase shifts (as in the activity "Changing Sines," also associated with this episode) to better understand the intricate designs associated with guilloche, as well as other applications.

Additional Resources

There are examples of directed practice in graphing some trigonometric by hand, followed by visual and audio examples, at:
http://www.mste.uiuc.edu/courses/ci399su01/students/aleonard/finalexam/part1.html

For a collection of examples of guilloche on currency and other documents, see:
http://www.maa.org/editorial/mathgames/mathgames_02_09_04.html

For the Student

This activity can help students visualize trigonometric identities. For example, $\sin(2x) = 2\sin(x)\cos(x)$ is a trigonometric identity, which means the equation is true for all values of x in the domain. Use a graphing calculator to graph each side of the equation as a separate function. The graphs appear to be identical, but the graphs do not constitute a proof. Another approach is to consider $y = \sin(2x) - 2\sin(x)\cos(x)$. If this an identity, the value of y should be 0 for all values of x, so the graph is just the x-axis.

After viewing the graphs, prove these identities. Find more identities in any trigonometry book and graph and prove them. There is a lesson on proving these at **http://kkuniyuk.com/M1410502.pdf**.

For an animated explanation of adding of sine waves, see:
http://www.purchon.com/physics/sines.htm

Related Topic

One of the applications of adding sine waves is what is called "interference" in light, electricity, etc. For more on this, see:
http://www.pbs.org/wgbh/nova/teachers/activities/pdf/2913_volcano_02.pdf
or
http://www.umanitoba.ca/faculties/arts/linguistics/russell/138/sec4/acoust1.htm

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