Shortest Path

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Abstract: This activity is an application of definite integrals. Students first graph three functions on the same domain. Each of these goes through the same three points. They then find the shortest of each of the paths through these points using the symbolic capacity of their calculator and calculus. Finally they explore a piece-wise linear function which also goes through the three points. In this process they explore the arc length formula.

NCTM Principles and Standards:
Algebra standards
a) Understand patterns, relations, and functions
b) generalize patterns using explicitly defined and recursively defined functions;
c) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
d) use symbolic algebra to represent and explain mathematical relationships;
e) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
f) draw reasonable conclusions about a situation being modeled.

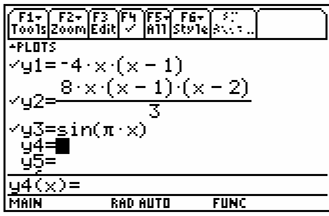
Geometry standards: Analyze characteristics and properties of two- and three-dimensional geometric shapes and mathematical about geometric relationships
Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.
Reasoning and Proof Standard
a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

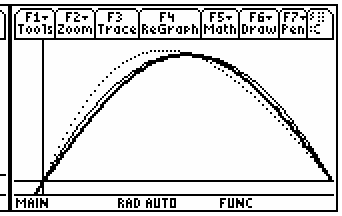
Representation Standard : use representations to model and interpret physical, social, and phenomena.

Key topic: Applications of definite integrals, arc length formula.

Degree of Difficulty: moderate
Needed Materials: TI-89 calculator

Situation: Consider the graphs of $y = -4x(x-1)$, $y = 8x(x - 1)(x - 2)/3$, and $y = \sin(\pi x)$. Graph them using the domain [0, 1]





Based on the graphs complete the table:

Function	Left x-intercept	Right x-intercept	Coordinates of maximum point
y = -4x(x-1)			
y = 8x(x - 1)(x - 2)/3			
y = sin(πx)			

Note: Each of the answers is:

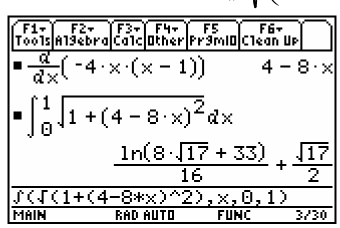
(0, 0)	*(1, 0)*	*(1/2, 1)*

These three functions go through many of the same points and have, on this domain, similar values and graphs.

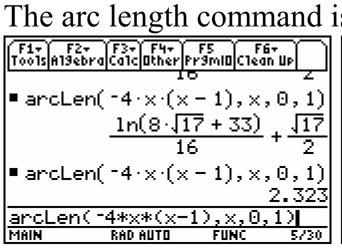
Which of these is the shortest curve going through the three points? To answer this question, find the length of each of these graphs on the interval [0, 1]. The TI-89 has a built in arc length function, but before we use this, let's review the arc length formula:

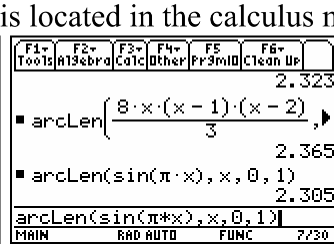
The length of the curve y = f(x) on the interval [a, b] is given by: $\int_a^b \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}dx$

We can apply this formula for our first function:



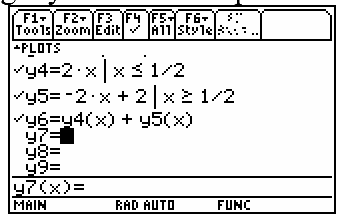
The arc length command is located in the calculus menu:

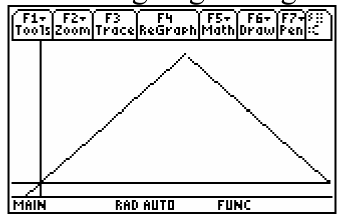


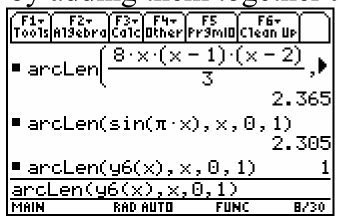


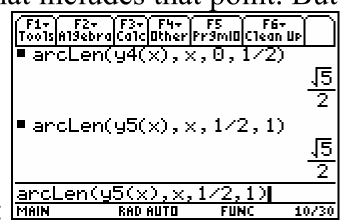
If the calculator can perform the anti-differentiation, it will produce an exact answer - otherwise it will give a decimal approximation.

The shortest path, of all three, is given by the function: y = sin(πx).
Construct a function which will give the shortest possible path connecting the three points: (0, 0), (1/2, 1) and (1, 1). Answer: That path must be made of line segments.

The equation of the line going through the first two points has a slope of 2 and a y-intercept of 0, so we get y = 2x. The equation of the line going through the other two points is y = -2x + 2. 



We can create a combination of these two line segments by adding them together as shown in y6. Let's find the arc length of our new function: 

Is this correct? No- the length must be considerably larger than 1. Why did the calculator produce an erroneous result? The reason is that our function y6(x) is not differentiable for all points in the domain [0, 1] - and in particular it is not differentiable when x = ½ and therefore we cannot use the arc length formula on any interval that includes that point. But we can, however, use it on each component of the function: 

The two segments have a total length of $\sqrt{5} \approx 2.236$. This number, then, is the limiting value of the length of a curve that goes through the three points. Can you find a differentiable function which passes through the points (0, 0), (1/2, 1) and (1, 1) whose length is shorter than that of y = sin(πx)?