Law of Cosines

MATH NSPIRED

TEACHER NOTES

Math Objectives

- Students will be able to state the Law of Cosines.
- Students will be able to apply the Law of Cosines to find missing sides and angles in a triangle.
- Students will understand why the Law of Cosines is true.
- Students will use appropriate tools strategically (CCSS Mathematical Practice).

Vocabulary

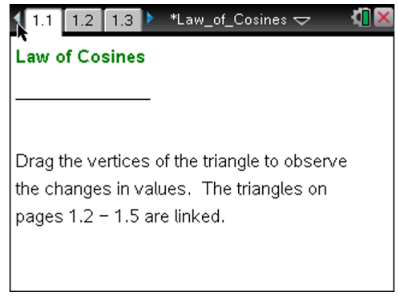
- cosine of an angle
- obtuse angle
- acute angle
- right angle

About the Lesson

- This lesson involves visualizing and exploring the Law of Cosines.
- Note: Some portions of the activity require CAS functionality – TI-Nspire CAS Required.
- As a result, students will:
  - Manipulate a triangle to observe the equality of the Law of Cosines.
  - Manipulate a triangle to discover why the Law of Cosines is true.
  - Consider the Law of Cosines in connection with previous knowledge about right triangle trigonometry.
  - Determine when the Law of Cosines can be used to solve for unknown side lengths and angle measures in a triangle, and apply it to solve when possible.

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to examine patterns that emerge.
- Use Quick Poll to assess students' understanding.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- Once a function has been graphed, the entry line can be shown by pressing ctrl G. The entry line can also be expanded or collapsed by clicking the chevron.

Lesson Files:

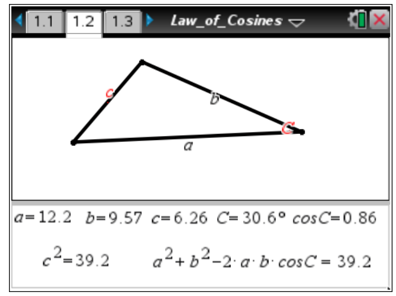
Student Activity
Law_of_Cosines_Student.pdf
Law_of_Cosines_Student.doc

TI-Nspire document
Law_of_Cosines.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

Move to page 1.2.



1. This page shows triangle *ABC*, with angles *A*, *B*, and *C*, and corresponding sides opposite those angles whose lengths are *a*, *b*, and *c*, respectively. You can drag any of the vertices to change the triangle.
   a. Below the triangle, you see the measures of two sides, an angle, the cosine of the angle, and two equations. Describe in words what the expression $a^2 + b^2 - 2ab\cos C$ tells you about the triangle.

   Sample Answers: The expression $a^2 + b^2 - 2ab\cos C$ gives the sum of the squares of the lengths of two sides of the triangle minus twice the product of those side lengths with the cosine of the included angle.

   b. Drag the vertices and observe the values of the two expressions at the bottom of the screen. What do you observe?

   Sample Answers: The values of the two expressions are equal for all possible triangles.

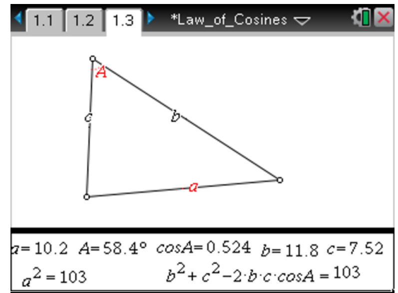
TI-Nspire Navigator Opportunity: ***Screen Capture***
See Note 1 at the end of this lesson.

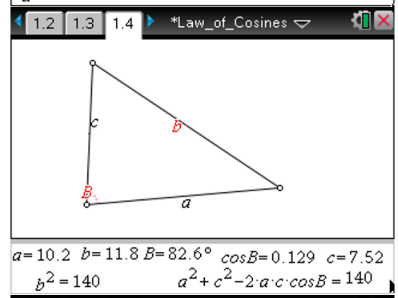
   c. Do you think this relationship will hold for the other sides and angles? For example, if you switch side *a* with side *c* and angle *A* with angle *C*, will the equation $a^2 = b^2 + c^2 - 2bc\cos A$ be satisfied? Explain your reasoning.

   Sample Answers: Yes. The labeling of the triangle does not affect the relationships. For example, one could re-label angle *A* as *C*, and angle *C* as *A*, changing the corresponding labels of the sides, and the equality expressed would hold.

TI-Nspire Navigator Opportunity: ***Quick Poll***
See Note 2 at the end of this lesson.

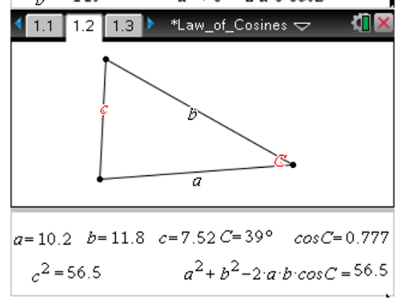
d. **Move to page 1.3 and to page 1.4** to test your prediction in part c. Drag the vertices to observe the values of the two equations. Why do you think this relationship might hold?





Sample Answers: Student answers will vary. Some students might observe that it appears to be connected to the Pythagorean Theorem.

Move back to page 1.2.



2. Adjust the triangle so that the measure of angle *C* is 90°.
   a. What is the cosine of angle *C*? How do you know?

   Sample Answers: The cosine of angle *C* is 0. Most students will have this memorized, though some might explain using the unit circle or the graph of the cosine function.

TI-Nspire Navigator Opportunity: *Quick Poll*   **See Note 3 at the end of this lesson.**

> **Teacher Tip:** The calculation of the cosine might show a very small, but non-zero number. Teachers might want to ensure that students understand this is due to rounding within the calculation, and should use their knowledge of the cosine function to answer this question.

   b. For any triangle with the measure of angle *C* = 90°, what is $a^2 + b^2 - 2ab\cos C$? How do you know?

   Sample Answers: It is always $a^2 + b^2$. Since $\cos C = 0$, we have $2ab\cos C = 0$. Therefore, $a^2 + b^2 - 2ab\cos C = a^2 + b^2 - 0 = a^2 + b^2$.

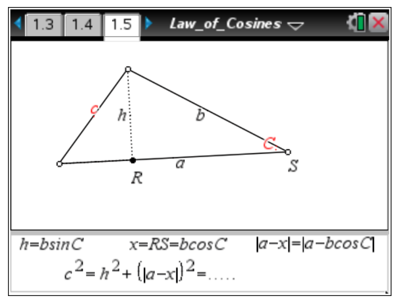
c. Why must it be true, in this case, that $c^2 = a^2 + b^2 - 2ab\cos C$?

Sample Answers: Since the triangle has right angle *C*, the Pythagorean Theorem gives $a^2 + b^2 = c^2$, and part b shows that $a^2 + b^2 - 2\,ab\,\cos C = a^2 + b^2$. Therefore, $c^2 = a^2 + b^2 - 2ab\cos C$.

d. What would the equality in part c be if angle *A* were the right angle? Angle *B*? Explain.

Sample Answers: If angle *A* were the right angle, the equality would be $a^2 = b^2 + c^2 - 2bc\cos A$. If angle *B* were the right angle, the equality would be $b^2 = a^2 + c^2 - 2ac\cos B$. In each case, this is simply the Pythagorean Theorem, as the cosine of the right angle will always be 0.

Move to page 1.5.



3. The equality you just showed in question 2 is called the Law of Cosines. It is true for all triangles, not just right triangles. On Page 1.5, investigate why the Law of Cosines is true.

   a. Move a vertex so that angle *C* is acute. The segment *h* is called an altitude of the triangle, and it is perpendicular to the side it intersects. Explain why the statement at the bottom of the screen, $h = b\sin C$, is true.

Sample Answers: The perpendicular makes a right triangle with side lengths *h*, *b* and *RS*. Then by triangle trigonometry, $\sin C = \frac{h}{b}$, so $h = b\sin C$.

   b. Explain why the statement at the bottom of the screen, $RS = |b\cos C|$, is also true.

Sample Answers: Again, using triangle trigonometry, $\cos C = \frac{RS}{b}$, so $RS = b\cos C$, and thus $RS = |b\cos C|$.

   c. Explain why the statement at the bottom of the screen, $c^2 = h^2 + (|a - x|)^2$, is true.

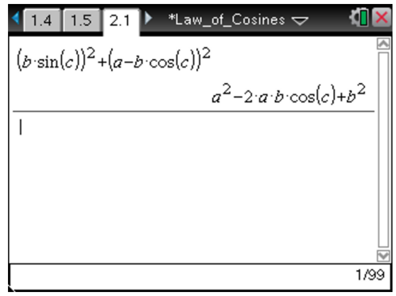
Sample Answers: Dropping the perpendicular h makes a second right triangle, the triangle with side lengths *c*, *h*, and *a* – *RS* = $a - x$. Thus, by triangle trigonometry, $c^2 = h^2 + (|a - x|)^2$.

   d. Rewrite the statement in part c using only the sides and angles of the original triangle.

<u>Sample Answers:</u> $c^2 = a^2 + b^2 - 2ab\cos C$

> **Teacher Tip:** Students might not square the terms, and so might have the response $c^2 = (b\sin C)^2 + (a - b\cos C)^2$. They will square the terms in the next question, so there is no need to push them to do so in this problem.

Move to page 2.1.



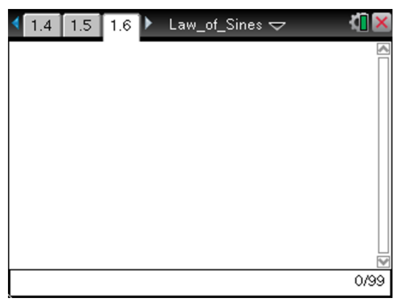
4. Test your equality in question 3d by multiplying out $(b\sin C)^2 + (a - b\cos C)^2$.
   a. Multiply out the expression above in two ways: with the handheld and by hand. Compare the results. Does the handheld differ from your results? If so, how, and why?

<u>Sample Answers:</u> Student answers will vary. The most common variation will likely be $b^2\sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C$. This difference is due to the handheld's use of the Pythagorean Identity to simplify the expression.

   b. If you need to, use the handheld to help you reconcile the two results. What important identity is the handheld using in its calculations?

<u>Sample Answers:</u> The Pythagorean Identity. Because $\sin^2 C + \cos^2 C = 1$, $b^2\sin^2 C + b^2\cos^2 C = b^2$. This simplifies the expression to $a^2 + b^2 - 2ab\cos C$.

Move back to page 1.5.

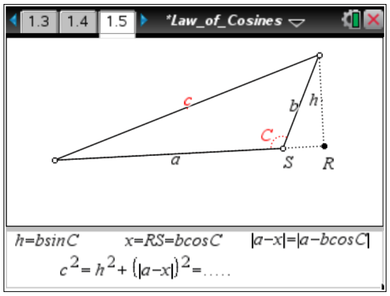


5. Move one or more vertices so that angle *C* is obtuse.
   a. What happens to *h*?

<u>Sample Answers:</u> It moves outside of the triangle.

> **Teacher Tip:** Students might have already observed the side of length *h* outside of the triangle in problem 3. However, the question could still be answered using the same arguments. In the case of obtuse *C*, the orientation of *h* with respect to *C* requires a different approach.

b. You might notice that the statements $h = b \sin C$ and $RS = |b \cos C|$ still appear on the screen. Show why they are both still true.



Sample Answers: Consider the triangle formed by the sides of length *RS*, *b*, and *h*:
The angle adjacent to side *RS* is supplementary to angle *C*.
Thus $\sin(180 - C) = \frac{h}{b}$, so $h = b\sin(180 - C)$.

But, by the properties of the sine function, $\sin(180 - C) = \sin C$, so $h = b \sin C$. Furthermore, $\cos(180 - C) = \frac{RS}{b}$, so $RS = b\cos(180 - C)$. By the properties of the cosine function, $\cos(180 - C) = -\cos C$, so $RS = -b\cos C$, and thus $|RS| = b\cos C$.

c. Is the Law of Cosines true when angle *C* is obtuse? Explain.

Sample Answers: Yes, the Law of Cosines is true when *C* is obtuse. In this case, we consider side *c* as the hypotenuse of the triangle with one leg of length *h* and another leg of length *a* + *x*. Then, by the Pythagorean Theorem, we have $c^2 = h^2 + (a + x)^2 = (b\sin C)^2 + (a - b\cos C)^2$ $= b^2 sin^2 C + a^2 - 2ab\cos C + b^2 cos^2 C = a^2 + b^2 - 2ab\cos C$.

6. Use the Law of Cosines, if possible, to solve for the missing side lengths and angles in each of the following triangles. If it is not possible to use the Law of Cosines, explain why not.

a. *a* = 5.8, *b* = 3.4, *C* = 64º

Sample Answers: *c* = 5.28, *A* = 81°, *B* = 35°

> **Teacher Tip:** Teachers might want to remind students to check the document settings and ensure that the document is set to measure angles using degrees.

b. *a* = 5, *b* = 8, *c* = 9

Sample Answers: *A* = 34°, *B* = 62°, *C* = 84°

c. *a* = 8, *c* = 12, *A* = 50º

Sample Answers: This problem cannot be solved with the Law of Cosines, as the Law of Cosines requires that two sides and the included angle, or three sides be known. In fact, no triangle with these specifications exists.

> **Teacher Tip:** Teachers might want to remind students of the conditions required to apply the Law of Sines and how to determine if one can build 0, 1, or 2 triangles with the specified dimensions.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- The Law of Cosines.
- That the Law of Cosines can be proved using triangle trigonometry and application of the Pythagorean Theorem.
- The conditions under which one can use the Law of Cosines to solve for unknown side lengths or angle measures in a triangle.
- How to apply the Law of Cosines to solve a triangle.

Assessment

Teachers can give students more triangles to solve. Teachers might want to include contextual problems to ensure that students experience the applications of the Law of Cosines.

TI-Nspire Navigator

Note 1

Name of Feature: Screen Capture

A Screen Capture can be used to show that for all the different triangles investigated by students, the Law of Cosines continues to hold.

Note 2

Name of Feature: Quick Poll

A Quick Poll can be used to determine if students think the relationship observed will hold for other angles in the triangle. This could generate a whole group discussion prior to the exploration.

Note 3

Name of Feature: Quick Poll

A Quick Poll can be used to ensure that all students understand that $\cos C = 0$.