

Applications of the Derivative

In this chapter, you will explore two common applications of the derivative, optimization and a related rate.

# Example 1: Designing a cylinder

Many optimization problems involve volumes and surface areas. This example shows how to solve a classic cylinder problem with the TI-89.

A right circular cylinder with a top has a volume of 355 ml. Determine the dimensions of the cylinder with minimal surface area.

#### Solution

Define the surface area as a function of the radius. Compute the first derivative, set it equal to zero, and determine the minimum point. You also can obtain the same result on a graph with the minimization commands.

### Solving numerically

- 1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
- 2. Store an expression for the volume to the variable *v*.

2nd  $[\pi] \times \mathbb{R} \land 2 \times \mathbb{H}$  STOF VENTER.

Store an expression for the surface area to the variable sa in a similar manner.

2 [2nd] [ $\pi$ ]  $\times$  R  $\land$  2 + 2 [2nd] [ $\pi$ ]  $\times$  R  $\times$  H [STO• SA [ENTER]



3. Since the volume is constant for this problem, you can use the **solve(** command to solve for *h* in terms of *r*.

F21:solve( V = 355 , H ) ENTER

4. Substitute the result from step 3 into the surface area formula to express the surface area as a function of r only.

5. Compute and store the derivative of your surface area function.

 $[2nd [d] \bigcirc [ENTER], R ) [STO> DSA [ENTER]$ 

6. Use the **solve(** command to find the value of r when the derivative is zero.

F21:solve( ENTER = 0, R) ENTER

To see a decimal estimate for the value, press • ENTER.

F1+ F2+ F3+ F4+ ToolsA19ebraCalcOther ■π·r <sup>∠</sup> ·h → V	F5 F6+ Pr9mi0C1ean UP h · π · r <sup>2</sup>
■2·π·r <sup>2</sup> +2·π·r	∙h≯sa
<b>2</b> · π	·r++2·h·π·r
■ solve(v = 355,	h) $h = \frac{355}{\pi \cdot r^2}$
solve(v=355.h)	
MAIN RAD AUTO	FUNC 3/30
F1+ F2+ F3+ F4+	F5 F6+
Tools Algebra Calc Other	Pr9MIDICIean UP
• solve(v = 355,	n) $n = \frac{1}{\pi \cdot r^2}$
• sa   h = $\frac{355}{\pi \cdot n^2}$	
	2 710
	$2\cdot\pi\cdot r^2 + \frac{110}{r}$
salh=355/(π*r^2	2)
MAIN RAD AUTO	FUNC 4/30
F1+ F2+ F3+ F4+ ToolsAl9ebraCalcOther	FS F6+ Pr9mIOClean Up
	$2 \cdot \pi \cdot r^2 + \frac{710}{7}$
	r
$= \frac{\alpha}{2} \left[ 2 \cdot \pi \cdot r^2 + \frac{7}{2} \right]$	$10$ $\rightarrow dsa$
art	r )
	$4 \cdot \pi \cdot r = \frac{710}{2}$
	r <sup>2</sup>
d(2*π*r^2+710∕r	r,r)→dsa
MHIN KHO HUTO	FUNC 5730
[F1+] F2+ [F3+] F4+]	F5   F6+
Too1sA19ebraCa1cOther	Pr9ml0 Clean Up
	r~
solve 4·π·n	$\frac{710}{5} = 0.r$
	r <sup>2</sup> "")
	3551/3.22/3
r =	0.00 2
	2·π
MAIN RAD AUTO	FUNC 6/30
[F1+] F2+ [F3+] F4+	F5 F6+
Tools Algebra Calc Other	Pr9mIO Clean Up
	3551/3.22/3
r=	2·π <sup>1/3</sup>
. (.	710 )
solve 4·π·r-·	$\frac{1}{2} = 0, r$
	r - )
coluo(4*π*n−71)	$r^{-} = 3.03722$
20106(4*)[*P-710	2-0,17

7. To test that the point is a minimum, you can compute the value of the derivative at a point on each side. First, test a point on the left.

DSA || R = 3.8 ENTER

Now test a point on the right.

● ← 9 ENTER

The derivative values imply that the curve is falling and then rising, so your value is a minimum point.

8. Alternately, you can use the second derivative test to test that the point is a minimum. There are two ways to compute the second derivative. Since **dsa** is the derivative of **sa**, you can differentiate **dsa** and evaluate it at the value of *r* computed as the critical point in step 6.

You also can compute a second derivative of **sa** at the same critical point. Recall that a second derivative is computed when a 2 is used as the third argument of the differentiate command.

9. Compute the height of the cylinder using the value of r from step 6.

355  $\div$  ( 2nd [ $\pi$ ] R  $\land$  2 ) [  $\odot$  (11 times) ENTER ENTER

F1+ F2+ F3+ F4+ F5 ToolsAl3ebraCalcOtherPr3mlDClean Up			
$2 \cdot \pi^{1/3}$			
- 5010e(4.), r r	$\frac{2}{r} = 3.83722$		
■dsa r=3.8	-1.41677		
dsalr=3.8 Main Radiauto	FUNC 8/30		
F1+ F2+ F3+ F4+ F5 F6+ ToolsAl9ebraCalcather Pr9miDClean Up			
$=$ solve $\left[4 \cdot \pi \cdot r - \frac{7}{r}\right]$	$\left[\frac{10}{2}=0,r\right]$		
∎destr=₹ 8	r = 3.83722		
■ dsa   r = 3.9	2.32903		
dsair=3.9 Main Rabiauto	FUNC 9/30		
F1+ F2+ F3+ F4+ Tools #19 sbr al calc lither lev	F5 F6+		
• dsa   r = 3.8	-1.41677		
■dsa r=3.9	2.32903		
$\left  \frac{d}{dr} (dsa) \right  r = 3.8$	3721524801		
dsa,r)lr=3.8372 Main Radiauto	37.6991 152480156 FUNC 10/30		
F1+ F2+ F3+ F4+ ToolsAl9ebraCalcOtherP	F5 F6+ r9ml0Clean Up		
$\frac{d^2}{d^2}(sa) r=3.3$	8372152480 ►		
dr <sup>2</sup> 1	37.6991		
<mark>¦2(sa,r,2) r=3.8</mark> 3 Main RaDauto	3721524801		
	FUNC 1730		
F1+ F1+ F1+ F1+ F1+ Tools \$3.4.5mg(ss., 025.4 Pt	F5 5		
F1+ Tools(\$1.5***(00.000)) • dsa   n = 3.8	F5 5 5 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7		
F1+ 5:- 130 130 100 Pr = dsa   r = 3.8 = dsa   r = 3.9	FINE 1/30		
$\frac{F_{10015}}{10015} = \frac{1}{3} + $	Find      Size        Find      Size      Size        F=3.83722      -1.41677      2.32903        Size      Size      Size        Size      Size      Size		
$ \frac{f_1}{10015(k_1,k_2,m_1)} = \frac{1}{10015(k_1,k_2,m_1)} = \frac{1}{10015(k_1,k_2,m_1)} = \frac{1}{10005(k_1,k_2,m_1)} = \frac{1}{10005(k_1,k$	Fill      Fill        Fill      Similar        FUNC      \$7/10		
$ \frac{f_{1}}{Tools} = \frac{1}{2} \frac{f_{1}}{f_{1}} \frac{f_{2}}{f_{2}} \frac$	FUNC      1/30        F5		
$ \frac{F_{1}}{Tools} \frac{1}{s_{1}} \frac{1}{s_{2}} \frac{1}{s_{2}}$	FUNC      1/30        75      1.41677        2.32903      3721524801)        FUNC      1/40        FUNC      1/40		
$\frac{f_{1}}{Tools} = \frac{1}{2} + \frac{1}{2$	FUNC      1/30        F5      5        F5      5        F5      5        F5      5        F1      41677        2.32903      3721524801        FUNC      4/10        FUNC      4/10        F5      5        37721524801      5        37721524801      37.6991        2152480154      5		
$\frac{F_{1}}{Tools} \frac{1}{s_{1}} \frac{1}{s_{2}} $	FUNC      1/30        75mill      55mill      55mill        1.41677      2.32903        3721524801      57mill        FUNC      1/40        FUNC      1/40        FUNC      1/40        75mill      57c1524801        37721524801      37c6991        2152480156      7c67443		

### Solving graphically

 Press ● [Y=] to display the Y= Editor. Press CLEAR as necessary to delete any functions. Define the function (the surface area formula from step 4 of *Solving Numerically*).

2 2nd [π] X ∩ 2 + 710 ÷ X ENTER

- 2. Press [WINDOW] and set the Window variable values as shown.
- 3. Press [GRAPH] to graph the function.
- 4. To compute the minimum point, press **F5 Math** and select **3:Minimum**. Now use **()**, or type a value to the left of the minimum point, and press **(ENTER)**. Press **()** or type a value for the right bound. The coordinates of the minimum point are displayed.

5. To compute a minimum on the Home screen without using derivatives, press HOME to return to the Home screen. Enter the command:

F3 6:Min(Y1 ( X ) , X ) ENTER



To see a decimal estimate for the value, press  $\bigcirc$  [ENTER].

F1+ F2+ Too1s A19ebi	raCalcOtherF	FS Fé r9ml0Clea	it n Up
~ `		7.	67443
■fMin(y	1(x), x		
× = -	<u>355</u> 1/3.2 2·π <sup>1/3</sup>	2/3 3 or	× = 0
■fMin(y	1(x), x)	×=3.	83722
fMin(y1 Main	(x),x) RAD AUTO	FUNC	13/30

## **Example 2: Related rates**

Many related rates examples investigate how the rates of change of two quantities are related. This classic sphere problem is done by considering both the volume and the radius as functions of time.

A spherical balloon is being inflated so that the radius is increasing at a steady rate of 2 cm/sec. Find the rate of change of the volume at any time t, and at the time when the radius is 8 cm.

- 1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
- 2. To define a volume function, press F4 **Other** and select **1:Define**. Enter the volume function as shown. Note that the radius is actually a function r(t) and that two sets of parentheses are needed to cube the radius.
- Compute the derivative.
  2nd [*d*] V ( T () , T () ENTER
- 4. Substitute 2 for  $\frac{dr}{dt}$  in the previous result to find the volume for *t*.

○ ENTER [] 2nd [d] R ( T ) , T ) = 2 ENTER

5. Use the previous result to find the volume when the radius is 8 cm.

○ ENTER 1 R ( T ) = 8 ENTER

F1+| F2+ | F3+| F4+ | F5 | F6+ Tools|A19ebra|Ca1cl0ther|Pr9ml0|Clean Up  $\pi \cdot (r(t))$ Define  $\frac{\alpha}{dt}(v(t))$ F1- F2-Tools Algebr (r(t)) 512 (t) = 8

# Exercises

Exercises 1 to 4 involve a right circular cylinder with no top that is constructed from 100 square cm of material.

- 1. Determine the volume as a function of the radius.
- 2. Compute the derivative of the volume function.
- 3. Use the derivative to determine the dimensions of the cylinder with maximum volume.
- 4. Compute the dimensions of the cylinder with maximum volume directly, without use of the derivative.

Exercises 5 and 6 involve a spherical iceball that is melting in such a way that the volume decreases at the rate of 6 cm<sup>3</sup> / sec.

- 5. Compute the derivative  $\frac{dr}{dt}$  at any time *t*.
- 6. Compute the derivative  $\frac{dr}{dt}$  at the time when the radius is 1 cm.