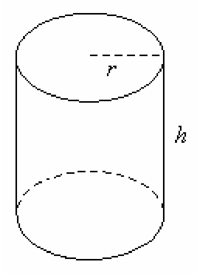
Ages 17-19 – Optimisation

1. A standard example

As an introduction, do a)-h) without using CAS



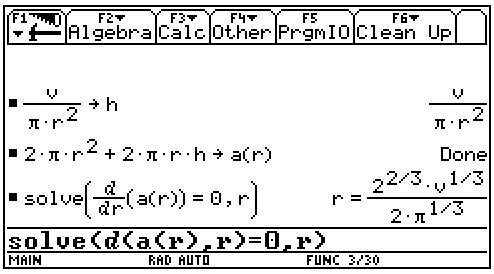
The cylinder shown has the volume 1.0 litre = 1 dm^3.

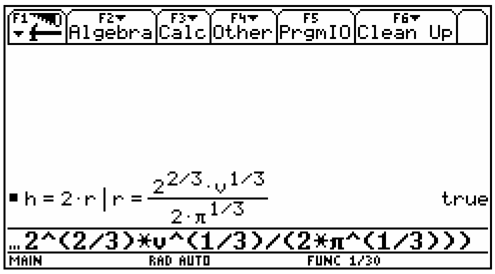
We wish to determine the radius *r* and height *h* that gives the minimum surface area of the cylinder.

a) Write the volume $V = 1$ dm^3 of the cylinder in terms of *h* and *r* .
b) Solve the equation in (1) for *h* .
c) Write the surface area *A* of the cylinder (side+bottom+top) in terms of *h* and *r* .
d) Substituting *h* from (3), write *A* as a function of *r* .
e) Determine the derivative $A'(r)$.
f) Solve the equation $A'(r) = 0$ for *r*.
g) Calculate the corresponding value of *h* by substituting the value of *r* just found into (3) .
   Note: The unit of *h* and *r* is dm.
h) Compare the numerical values of *h* and *r*. Conjecture? Prove the conjecture.

Solution (partial)

The numerical values found for the special case *V* = 1.0 litre suggest the conjecture that the minimum area is obtained when height = diameter. Of course it is possible to prove the conjecture algebraïcally using paper and pencil methods only. Below we show how to do it with CAS.





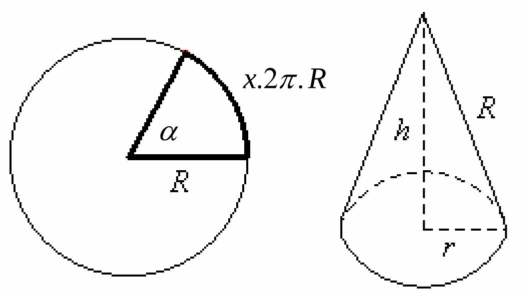
Exercise

Consider a similar problem for a cylinder without a top.

(i) Determine the dimensions giving the minimum surface area for a volume of 1.0 litre.
(ii) Compare height and radius for any volume *V* where the cylinder has a minimum surface area.

Answer to b: height = radius.

2. Make a cone



A section is cut from a circular piece of paper, and folded to make a cone. The radius of the paper is *R*. You now remove a fraction *x* of the paper. If for example *x* = 0.25, the centre angle α of the section cut out for the cone is 90°. The base radius of the cone is given by *r* and the height by *h* .

On a Voyage 200 we can write `rr` for *R*, to distinguish between the base radius of the cone and the radius of the circular piece of paper from which the cone is made.

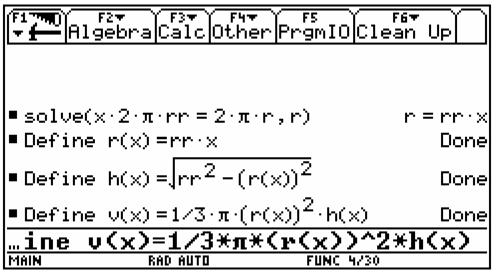
a) Determine *r* and *h* in terms of *x* and *R* and find the volume *V*(*x*) of the cone in terms of *x* and *R*

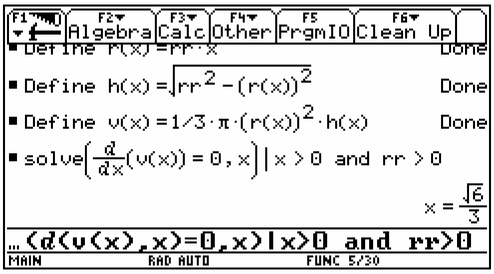
b) Determine $V'(x)$ and solve the equation $V'(x) = 0$.

c) Calculate the fraction *x* and the corresponding angle α that gives the maximum cone volume

d) Repeat this problem with the centre angle α as the independent variable.

<u>Solution</u> (partial)





The maximum for $x = \frac{\sqrt{6}}{3} \approx 0.8165$ corresponds to the angle $\alpha \approx 293.9°$.

<u>Exercise</u>

Imagine that you fold a cone using the sector that was removed from the original circular piece of paper. The total volume of the two cones will be given by $f(x) = V(x) + V(1-x)$. Determine the maximum volume using non-CAS methods.

<u>Solution</u>

It is not easy to find the solution symbolically and voyage 200 fails to do it because it cannot handle cubic equations symbolically. By using a graphic or numerical tool we obtain the answer $x \approx 0.3240$ or $x \approx 0.6760$. It is difficult to see the maxima on the graph because *f* is "almost constant" when $0.25 < x < 0.75$. There is a relative minimum at $x = 1/2$.