

Limits, Numerically

by Dave Slomer

The sequences $a_n = \left(\frac{1}{10}\right)^n$ and $b_n = (10)^n$ are both *geometric*, but behave very differently. In normal graph mode, define $y1 = .1^x$ and $y2 = 10^x$. Set up the TI-89's [TABLE] to start at 1 and go up by 1. The table will look like so:

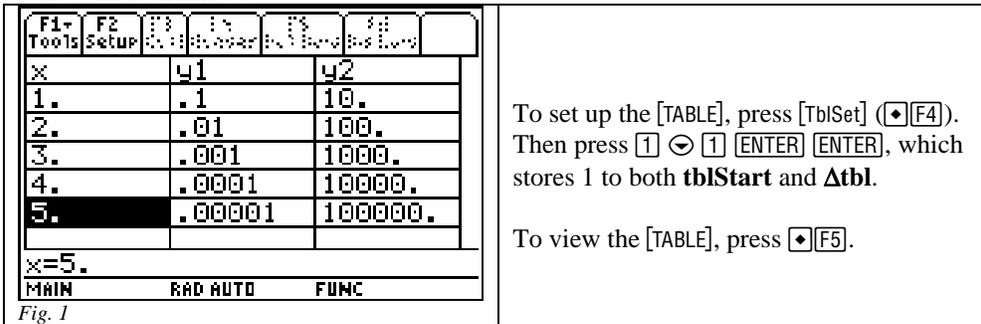


Fig. 1

You see $y1$ rapidly “shrinking” (toward 0) while $y2$ is just as rapidly “growing” (toward ∞). In calculus symbols:

$$\lim_{n \rightarrow \infty} \left(\frac{1}{10}\right)^n = 0 \text{ and } \lim_{n \rightarrow \infty} (10)^n = \infty.$$

Exercise 1: Find $\lim_{n \rightarrow \infty} \left(2 + \left(\frac{1}{10}\right)^n\right)$ and $\lim_{n \rightarrow \infty} \left(2 - \left(\frac{1}{10}\right)^n\right)$

In both limits, the fractional term approaches 0 while the first term is 2. Applying the “Limit of a sum is the sum of the limits” theorem, the limit is therefore 2 for both sequences, but the first approaches 2 “from above” while the second approaches 2 “from below.” Here’s a numerical look, where $y1 = 2 + (.1)^x$ and $y2 = 2 - (.1)^x$:

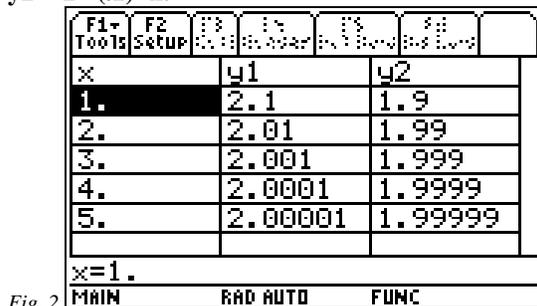


Fig. 2a

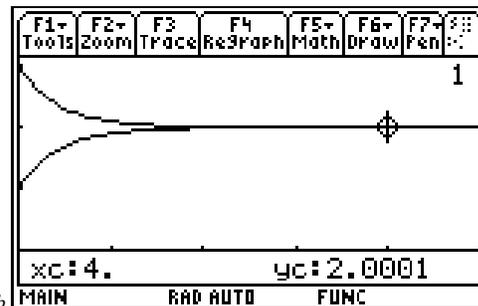


Fig. 2b

In both this exercise and the initial example it is important to notice that the independent variable is nowhere near ∞ . In both tables, the independent variable has only reached 5. Yet, already the sequence behavior is clear, so rapid is the approach behavior. Figure 2b gives a graphical look. Do the phrases “from above” and “from below” make more sense now?

Exercise 2: Find $\lim_{n \rightarrow \infty} \left(- (10)^n\right)$.

Algebraically, the “-” sign amounts to multiplying by -1 , a constant, which can be “factored” out of the limit expression, giving $-\lim_{n \rightarrow \infty} \left((10)^n\right)$. Hence, the limit is $-\infty$.

Exercise 3: Estimate the numeric value of $\lim_{x \rightarrow 2} \left(\frac{x-2}{x^2-4} \right)$.

This is not a sequence, but the same idea applies. The problem asks, what does $\left(\frac{x-2}{x^2-4} \right)$ approach as x gets closer and closer to 2, both through larger values (2.1, 2.01, 2.001, ...) and smaller (1.9, 1.99, 1.999...)? A numeric approach could be to **TRACE** the function's graph, and **ZOOM** in on points with x -coordinate near 2, though usually such coordinates are ugly (see figure 3b). Setting the TI-89 table to start at 2 and go up by a small amount (.1 or .01 or .001, etc.) and then moving the cursor up in the table past 2 to get a table such as the one below can be better, especially if combined with the graph:

x	y1
1.9998	.2500125
1.9999	.2500063
2.	undef
2.0001	.2499938
2.0002	.2499875

x=1.9998

Fig 3

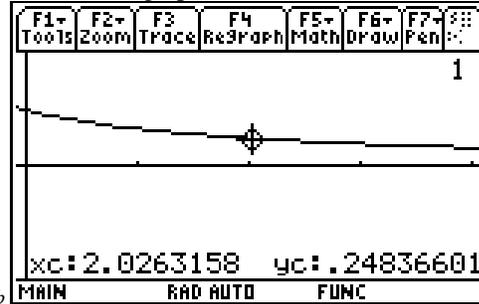


Fig 3b

With $\Delta tbl = .0001$, we see fairly strong evidence that the limit is 0.25, although the function is undefined at 2.

[Both pieces of information are important, however—the *behavior* of f near 2 appears to be that points “cluster” about the point (2, 0.25), but there is a “hole” in the graph at (2, 0.25), where f is *undefined*. (Confirm this by tracing to $x = 2$.) In all limit problems, we seek to describe the *behavior* of the function near, not necessarily at, a certain value. It is algebra, not calculus, that tells us that $f(2)$ is undefined. However, calculus tell us that f is *discontinuous* at $x = 2$ because of all of this.]

Exercise 4: Find an approximate numeric value for $\lim_{x \rightarrow \pi} \left(\frac{x-\pi}{x^2-\pi^2} \right)$, accurate to 5 decimal places.

We'll just look at a table, since the graph looks virtually the same as the one for Exercise 3 [the π 's just replace three 2's]. But trying the same approach as in Exercise 3 leaves us wondering. With $tblStart = \pi$ and $\Delta tbl = .0001$, we see the following table, after moving 2 rows up past π :

x	y1
3.141393	.15916
3.141493	.1591575
3.141593	.1538462
3.141693	.1591524
3.141793	.1591499

x=3.14139265359

Fig. 4

Not only is it difficult to see roughly where π is, it isn't even clear to three decimal places what the limit might be—what, exactly, is going on in that middle row?? We could make Δtbl smaller and smaller and play with the cursor keys for a while, but we need a better-organized, smoother approach than that. We need a way to make x approach π , both from the right (values larger than π) and left (values smaller than π). So, there is a “sub-exercise” to work on before we can efficiently finish this exercise.

Exercise 4b: Find a sequence that approaches π through larger values. Then find one that approaches π through smaller values.

In Exercise 1, we learned that $\lim_{n \rightarrow \infty} \left(2 + \left(\frac{1}{10} \right)^n \right)$ approached 2 through larger values and that

$\lim_{n \rightarrow \infty} \left(2 - \left(\frac{1}{10} \right)^n \right)$ approached 2 through smaller values.

So, for the same reasons, $\lim_{n \rightarrow \infty} \left(\pi + \left(\frac{1}{10} \right)^n \right)$ approaches π from above and

$\lim_{n \rightarrow \infty} \left(\pi - \left(\frac{1}{10} \right)^n \right)$ approaches π from below.

But how do we make x -coordinates on a graph approach π that way?

The answer lies in **PARAMETRIC Graph MODE**. We will aim for the most general solution, using **a** instead of π and **f** instead of the given function.

First, on the **HOME** screen, press **[F6]** (which is **2nd[F1]**) and then press **[1]** to clear all one-letter variables from your '89's memory. Then make the **[Y=]** screen look like so:



Fig. 5

Here, **x1** defines a sequence that will approach **a** through smaller values, while **x2** defines one approaching **a** through larger values. Meanwhile, **y1** is a function of the **x1** sequence values, while **y2** is a function of **x2**. All that remains is to define **a** to equal π and **f** to equal the given function, which we can easily do on the **HOME** screen.

...meanwhile, back at Exercise 4...

Store π into **a** and define **f** to be our function:

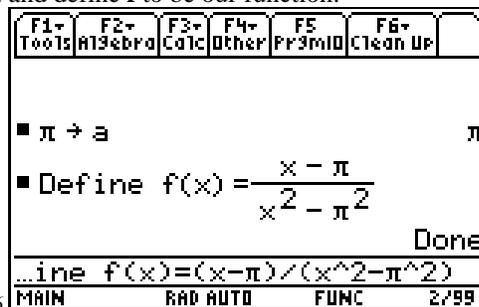


Fig. 6

To make **x** approach **a**, set up the table to produce **x**'s in a way similar to how we got the **x** values in Exercise 1—make **tblStart** = 1 and **Δtbl** = 1—always. For the function in Exercise 4, the table looks like this:

F1- Tools	F2- Setup	F3- Header	F4- Header	F5- Header	F6- Header
t	xt1	yt1			
1.	3.041593	.1617289			
2.	3.131593	.1594086			
3.	3.140593	.1591803			
4.	3.141493	.1591575			
5.	3.141583	.1591552			
yt1(t)=f(xt1(t))					
MAIN		RAD AUTO		PAR	

Fig. 7

F1- Tools	F2- Setup	F3- Header	F4- Header	F5- Header	F6- Header
t	xt2	yt2			
1.	3.241593	.1566616			
2.	3.151593	.158902			
3.	3.142593	.1591296			
4.	3.141693	.1591524			
5.	3.141603	.1591547			
yt2(t)=f(xt2(t))					
MAIN		RAD AUTO		PAR	

Fig. 7b

While x (that is, $xt1$) is fairly close to π , the function, f , appears to be approaching $.1592$, but we were asked for five-decimal-place accuracy, and we just don't see that yet. Not to worry—just use the cursor keys to move a few rows down through the table, closer and closer to π :

F1- Tools	F2- Setup	F3- Header	F4- Header	F5- Header	F6- Header
t	xt1	yt1			
5.	3.141583	.1591552			
6.	3.141592	.159155			
7.	3.141593	.159155			
8.	3.141593	.1591548			
9.	3.141593	.1591546			
yt1(t)=.15915457091928					
MAIN		RAD AUTO		PAR	

Fig. 8

F1- Tools	F2- Setup	F3- Header	F4- Header	F5- Header	F6- Header
t	xt2	yt2			
5.	3.141603	.1591547			
6.	3.141594	.1591549			
7.	3.141593	.159155			
8.	3.141593	.1591548			
9.	3.141593	.1591546			
yt2(t)=.15915457091928					
MAIN		RAD AUTO		PAR	

Fig. 8b

On the 9th row of the table, x is “very” close to π , and since the last entry on the screen—the one closest to π so far—starts with $.15915457$, it seems safe to go with $.15915$ as the estimate to 5 decimal places.

Exercise 5: Give numerical evidence that $\lim_{x \rightarrow 0} \left(\sin\left(\frac{1}{x}\right) \right)$ doesn't appear to exist.

On the **HOME** screen, define a and f :

F1- Tools	F2- Setup	F3- Header	F4- Header	F5- Header	F6- Header
0 → a					
Define f(x)=sin(1/x)					Done
Define f(x)=sin(1/x)					
MAIN		RAD AUTO		PAR	

Fig. 9

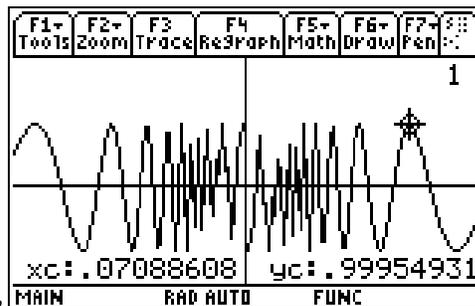


Fig. 9b

Looking at the table, even far down in the table, you will see no pattern—but by all means *look*. You will see rather indescribable, if not unpredictable, behavior. Since f doesn't appear to approach any particular number, we conclude that no limit exists. Such behavior may best be illustrated graphically via repeated **ZOOM In** at $(0,0)$. No matter how many times you zoom in (in theory), the graph always looks more or less like that in figure 9b. Viewed graphically, we realize that the function's behavior can be described: it is “oscillating wildly” between -1 and 1 as x approaches 0 .

Exercise 6: If $\lim_{x \rightarrow 0} \left(x \cdot \sin\left(\frac{1}{x}\right) \right)$ appears to exist, estimate its value. If, as in Exercise 5, you see no particular pattern, say that it probably does not exist and give reasons.

Exercise 7: Describe the “end behavior” of the *inverse tangent* function by estimating the numeric values of $\lim_{x \rightarrow \infty} (\tan^{-1} x)$ and $\lim_{x \rightarrow -\infty} (\tan^{-1} x)$.

Hmm... ∞ ...what to make **a** equal... ∞ ? (Your '89 will let you!) But, no, because, referring to the [Y=] screen in Exercise 4b, that would mean that values of the form $\infty \pm (.1)^t$ would need to be computed and that's just not going to work well. (Have your '89 compute $\infty - 1$, for example. Does the result surprise you?)

So what WILL work? Well, to make **x** approach ∞ quickly enough for the TI-89 table to help, we need **x** to take on values such as those values of **y2** in figure 1. And to make **x** approach $-\infty$, we'd need the opposites of those **y2** values.

So, define parametric functions 3 and 4 like so (and do not clear the first 2 parametric functions—just use [F4] to temporarily “turn them off”):

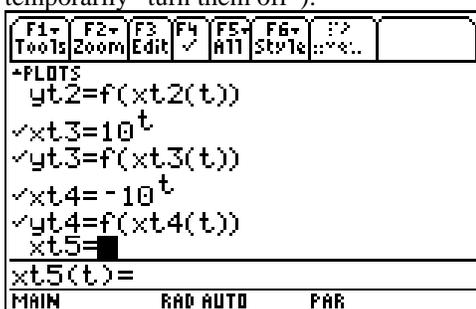


Fig. 10

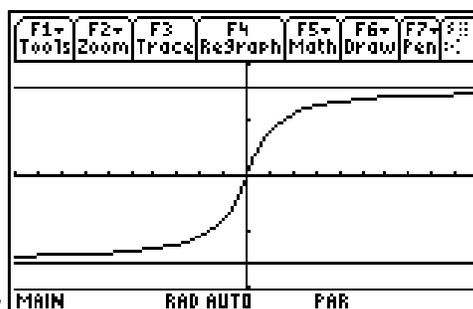


Fig. 11

Do you see that **xt3** will quickly approach ∞ , while **xt4** will rush toward $-\infty$? Now we can investigate the end behavior of **f** via **yt3** and **yt4**, each of which is **f** of **x**-coordinates generated by **xt3** or **xt4**.

The only thing left to do on the [HOME] screen is to **Define** $f(x)=\tan^{-1}(x)$, look at the table, and make a good guess based on the evidence.

The graph of this **f** is shown above in figure 10b, along with 2 horizontal lines—its *asymptotes*. Can you guess the exact equations of the asymptotes?

(Trial-and-error produced the [WINDOW] used, in which **x** goes from -10 to 10 , **y** from -2 to 2 , and **t** from -1 to 1 in steps of $.1$. Is it clear from the definitions of **xt3** and **xt4** why **t** has such a small range and why it has to include negative values?)

Example 8: Estimate $\lim_{n \rightarrow \infty} \left(\left(1 + \frac{1}{n} \right)^n \right)$ with accuracy to 6 decimal places. Can you guess the exact value of this limit?

Example 9: Estimate $\lim_{x \rightarrow \pm\infty} \left(\frac{\sin(x)}{x} \right)$ and $\lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right)$. Draw its graph to show the function's end behavior and its behavior near 0. (Note that the function is *odd* [Why?], which may save you some effort.)

In this activity, words and phrases such as “seems safe,” “appears,” “fairly strong,” and “very close,” permeate. It would be wrong to think that the work above is anything more than just wishful thinking. Whether obtained from graphs, one point at a time, or from tables, 5 at a time, the numeric evidence is very

strong, but it proves nothing. There are algebraic techniques, some easier than others, that can be applied to prove that each limit is exactly what it is. In fact, you should factor the denominator in Exercise 4, do a little algebra, and prove that the limit is exactly $\frac{1}{2\pi}$, which is approximately 0.159154, as we guessed.

Moreover, the TI-89 can do almost all of the required algebra via intrinsic routines and the [HOME] screen [F3] (Calculus) [3] (Limit) command. But understanding limits numerically is as essential as understanding them graphically or algebraically (symbolically). The three approaches help to form a seamless whole.

(You might want to save the 4 parametric functions used to help estimate the limits in this activity so you never have to type them again. To do so, on the [Y=] screen press [F1] [2] (Save copy as GDB—Graph DataBase) [down arrow] and type a name that will remind you of the activity and the fact that the name represents a Graph Database, such as **gdlimits**. Later, just opening the Graph Database will give you all 4 functions and set the mode to parametric. To do so, on the [Y=] screen press [F1] [1] [down arrow] and type the name you used earlier.)

Calculus Generic Scope and Sequence Topics: Limits and Continuity

NCTM Standards: Number and operations, Algebra, Geometry, Problem solving, Reasoning and proof, Connections, Communication, Representation