Investigating Properties of Lines in the Plane

Construct and Investigate:

1. Open a new Voyage™ 200 with Cabri screen and show the rectangular coordinate axes (◆ **F**). Draw a line attached to the origin of the axes that goes off to the upper right of the screen. Place points ***A*** and ***C*** on the line as shown in Figure 1. Construct a line parallel to the x axis through ***A*** and a second line parallel to the y axis through ***C***. Label the intersection of these two lines point ***B***, and hide the parallel lines. Construct and measure segments ***AB*** and ***BC***.

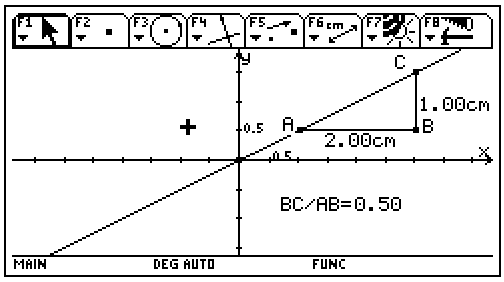

Figure 1

2. Using **Calculate**, compute the ratio $\frac{BC}{AB}$. Drag points ***A*** and ***C*** along the line, and explain what happens to the ratio $\frac{BC}{AB}$. Explain what happens to the ratio $\frac{BC}{AB}$ as you rotate the line around the origin.

3. The ratio $\frac{BC}{AB}$ is the ratio of $\frac{\textbf{Change in } y \textbf{ coordinates}}{\textbf{Change in } x \textbf{ coordinates}}$ and is called the **slope** of the line. Use the **Slope** measurement tool to show the slope of the line. Explain how the slope of the line compares with the ratio $\frac{BC}{AB}$ as you rotate the line around the origin. Are these two values always equal? Explain. Drag points ***A*** and ***C***, and explain how the slope is affected.

4. Using the **Equations and Coordinates** tool, find the equation of the line. Explain how this compares to the ratio $\frac{BC}{AB}$ and the slope as you rotate the line around the origin.

5. Place point ***D*** anywhere on the line, and show its coordinates. Explain the relationship between the coordinates of ***D***, the slope, and the equation of the line. Multiply the x coordinate by the slope. What do you get? Drag ***D*** along the line. Does the relationship remain the same? Explain. Rotate the line around the origin. Does the relationship remain the same? Explain.

6. Use the coordinates of points ***A***, ***B***, and ***C*** to explain the slope of a line in the coordinate plane.

Explore:

1. On a new Voyage 200 with Cabri screen, draw a line through the origin and show its slope using the **Slope** tool. The value of the slope can be positive or negative. The absolute value of the slope can be large or small. The slope can be zero, one, or infinitely large. Determine a way to organize the different slope values so you can quickly estimate the slope of any line.

2. On a new Voyage 200 with Cabri screen, place point ***A*** anywhere on the y axis and draw a line through this point. Show the coordinates of ***A***, the slope, and the equation of the line. Explain the relationship among the equation of the line, the coordinates of ***A***, and the slope of the line. Can you use the method you developed in part 1 of *Explore* to estimate the slope of any line in the plane? Explain.

Teacher’s Guide: Investigating Properties of Lines in the Plane

Construct and Investigate:

1. This construction creates a “slope triangle” attached to the line showing the relationship $\frac{\Delta y}{\Delta x}$, the definition of slope of a line.

2. Use the **Comment** tool to label the result of the computation as ***BC/AB* =** . As ***A*** and ***C*** are dragged along the line, the ratio $\frac{BC}{AB}$ remains unchanged. Measuring angles of $\triangle ABC$ helps convince students that no matter where points ***A*** and ***C*** are located, all versions of the triangle are similar to each other (Figure 2). Because similar triangles have proportional sides, the ratio $\frac{BC}{AB}$ stays constant as long as the line is not rotated.

   As the line is rotated around the origin, the angles of the slope triangle change, producing nonsimilar triangles (Figure 3). The ratio $\frac{BC}{AB}$ becomes larger as the line approaches a vertical position and smaller as it approaches a horizontal position.

3. The absolute value of the ratio $\frac{BC}{AB}$ and the slope are equal. The ratio $\frac{BC}{AB}$ gives only positive values for the slope because the two distances making up this ratio are always positive. The **Slope** measurement tool computes the slope correctly for all lines (Figure 4). As with the ratio $\frac{BC}{AB}$, the value of the slope remains unchanged as ***A*** and ***C*** are dragged along the line. The slope of a line is constant at all points.

4. The equation of lines through the origin are in the form $y = mx$, where m equals the slope of the line. As the line is rotated around the origin, the slope of the line is equal to the coefficient of x in the equation. When the line is neither horizontal nor vertical, the ratio $\frac{BC}{AB}$ is the same as the absolute value of the coefficient of x. This happens when the line passes through the second and fourth quadrants (Figure 5).

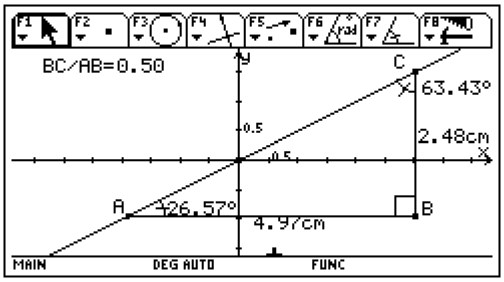

Figure 2

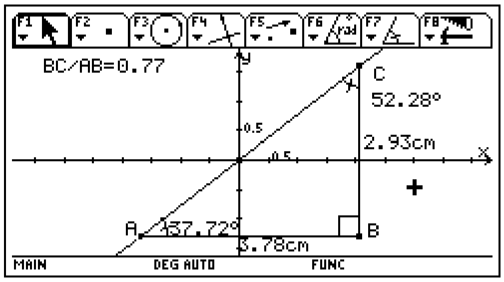

Figure 3

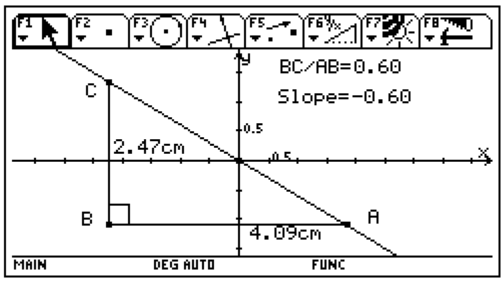

Figure 4

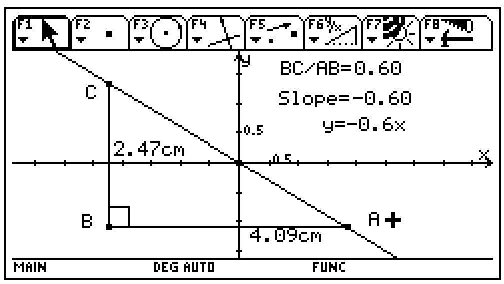

Figure 5

5. Point $\boldsymbol{D}$ represents any point on the line. Multiplying the x coordinate of point $\boldsymbol{D}$ by the slope of the line gives the y coordinate of point $\boldsymbol{D}$. This follows the algebraic equation $y = mx$, where m is the slope of the line. As $\boldsymbol{D}$ is moved along the line, the y coordinate is equal to the product of the slope m and the x coordinate of the point (Figure 6). This relationship remains true if the line is rotated around the origin. The algebraic definition $y = mx$ describes the y coordinate for any x coordinate on a line of slope m.

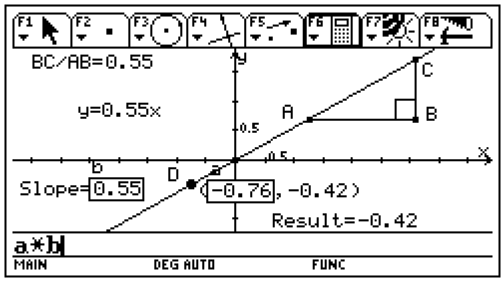

Figure 6

6. One purpose of this investigation is to have students connect the analytic definition of slope to the geometric definition you have been exploring. The length of $\overline{\boldsymbol{AB}}$ is the difference of the x coordinates of $\boldsymbol{B}$ and $\boldsymbol{A}$, or $(x_B - x_A)$. The length of $\overline{\boldsymbol{BC}}$ is the difference of the y coordinates of $\boldsymbol{C}$ and $\boldsymbol{B}$, or $(y_C - y_B)$. As shown in Figure 7, the ratio $\frac{\boldsymbol{BC}}{\boldsymbol{AB}}$ is equal to $\frac{y_C - y_B}{x_B - x_A}$. But the y coordinate of $\boldsymbol{B}$ is the same as the y coordinate of $\boldsymbol{A}$, and the x coordinate of $\boldsymbol{B}$ is the same as the x coordinate of $\boldsymbol{C}$. Therefore, the ratio $\frac{\boldsymbol{BC}}{\boldsymbol{AB}}$ can be written as the ratio $\frac{y_C - y_A}{x_C - x_A}$. This is the common definition of the slope when $m = \frac{y_1 - y_2}{x_1 - x_2}$ for any two points on the line.

   When $\boldsymbol{A}$ and $\boldsymbol{C}$ are dragged to different locations, or the line is rotated as shown in Figure 8, those relationships are still true. When the line is rotated so that the slope is negative, the sign of the ratio $\frac{y_C - y_B}{x_B - x_A}$ changes because the component $(x_B - x_A)$ becomes negative (Figure 9).

   Have students explore further to confirm that the slope of a line can be found using any two points on a line, in any order, as long as the coordinates are taken in pairs as in the equation $m = \frac{y_1 - y_2}{x_1 - x_2}$.

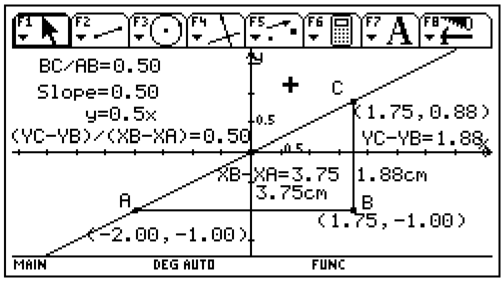

Figure 7

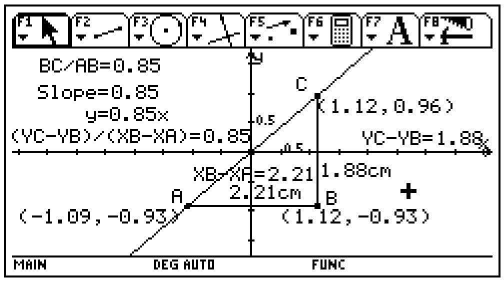

Figure 8

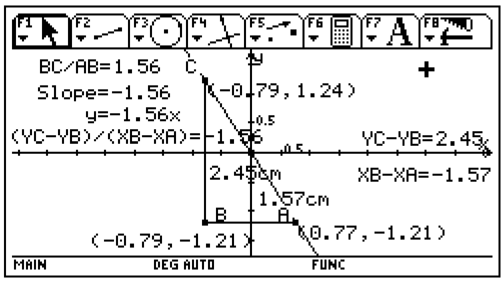

Figure 9

Explore:

1. Construct the lines $y = x$ and $y = -x$ to organize the different zones for estimating slope. There are four distinct zones:
   - When the line is between $y = x$ and the y axis, $m > 1$ (Figure 10).

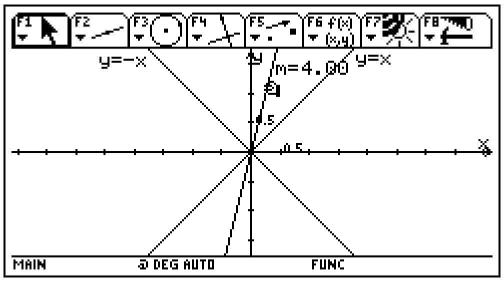

Figure 10

   - When the line is between $y = -x$ and the y axis, $m < -1$ (Figure 11).

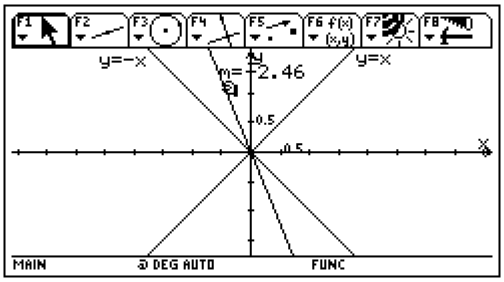

Figure 11

   - When the line is between $y = x$ and the x axis, $0 < m < 1$ (Figure 12).

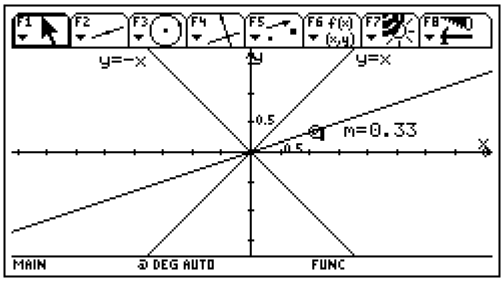

Figure 12

   - When the line is between $y = -x$ and the x axis, $-1 < m < 0$ (Figure 13).

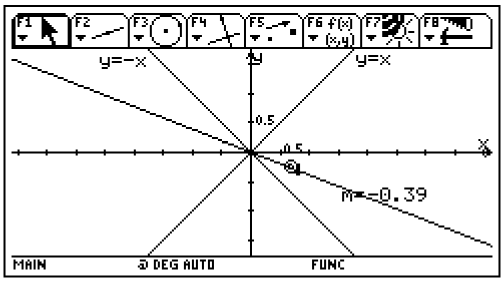

Figure 13

The line $y = x$ has a slope of 1, and the line $y = -x$ has a slope of -1 (Figures 14 and 15). When the line is on the x axis, the slope is zero (Figure 16). When the line lies on the y axis, the slope is infinite or undefined (Figure 17). If you hold ⇧ while rotating the line with ✋ locked, it moves in increments of 15° around the origin, stopping at 0° and 90° to show a zero slope and an infinite slope.

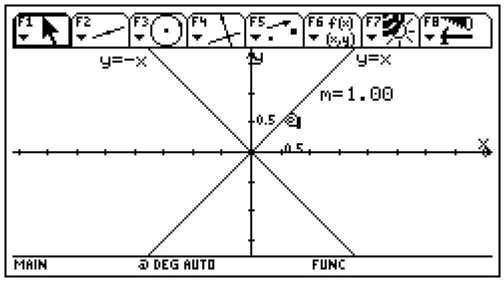

Figure 14

2. This exploration forms the basis for understanding the slope-intercept form of a linear function, $y = mx + b$. The slope of the line, m, is the coefficient of x. Point $\boldsymbol{A}$ has coordinate $(0, b)$ because it is on the y axis. The y coordinate of point $\boldsymbol{A}$ is the constant term b in the general slope-intercept model. As you rotate the line around point $\boldsymbol{A}$ or drag point $\boldsymbol{A}$ along the y axis, a match forms between the slope, the y coordinate of $\boldsymbol{A}$, and the equation of the line (Figure 18).

   The same method for estimating the slope developed in part 1 of *Explore* can be used for lines that do not pass through the origin. Draw (or imagine) a line parallel to the x axis through the y intercept. This line serves as a shifted x axis, making the y intercept of the line into the new "origin" (Figure 19). All of the zone relationships for lines through the origin apply to this new origin.

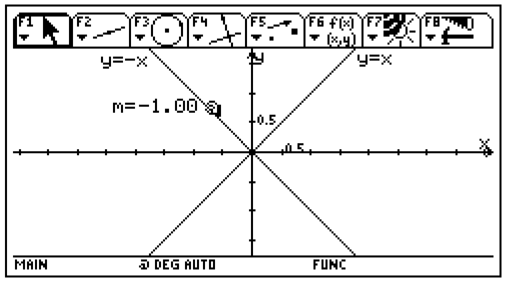

Figure 15

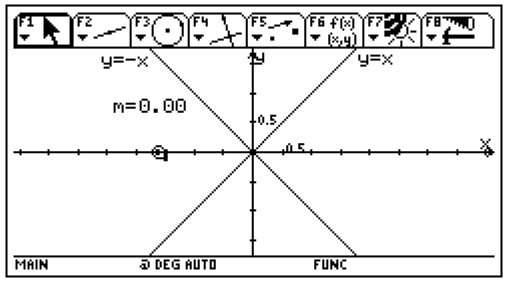

Figure 16

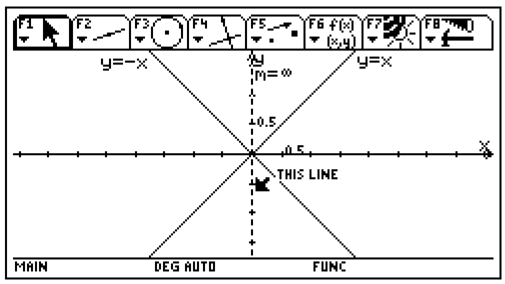

Figure 17

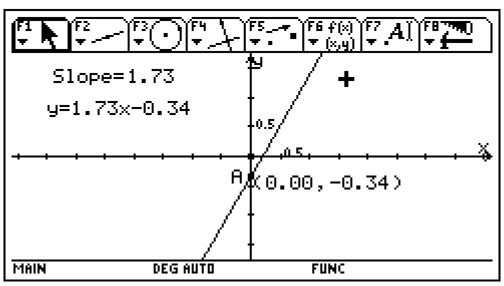

Figure 18

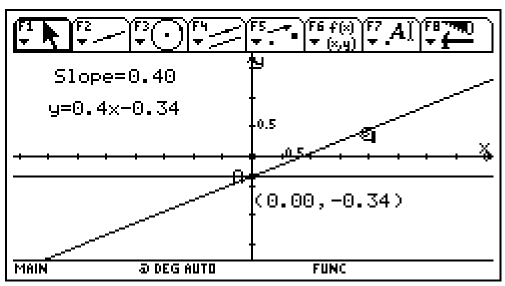

Figure 19