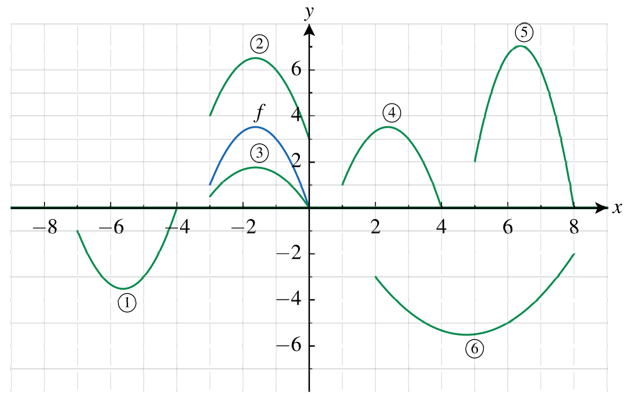
Thursday Night PreCalculus, October 26, 2023

New Functions from Old: Inverses, Transformations, and Compositions

Problems

1. The graph of f is given in the figure. Match each equation with its graph and give a reason for each choice.



(a) $y = f(x-4)$

Shift of the graph of $y = f(x)$ a distance 4 units to the right.

(b) $y = f(x) + 3$

Shift of the graph $y = f(x)$ a distance 3 units upward.

(c) $y = 2f(x-8)$

Shift of the graph of $y = f(x)$ a distance 8 units to the right.
A stretch of the graph of $y = f(x)$ vertically by a factor of 2.

(d) $y = \dfrac{1}{2}f(x)$

Shrink the graph of $y = f(x)$ vertically by a factor of 2.

(e) $y = -f(x+4)$

Shift of the graph of $y = f(x)$ a distance 4 units to the left. Reflect the graph across the x-axis.

(f) $y = -f(\frac{1}{2}(x-8)) - 2$

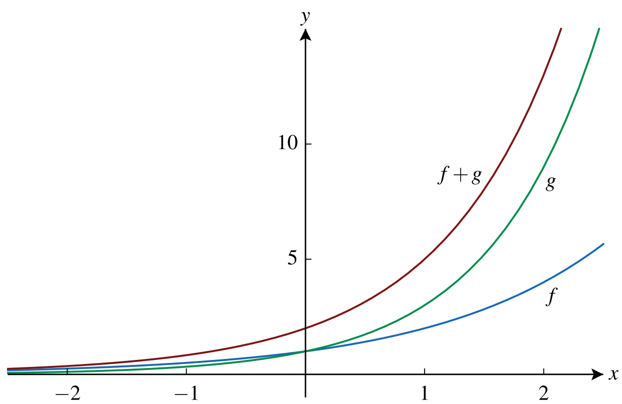
Shift the graph of $y = f(x)$ a distance 8 units to the right.
Stretch the graph horizontally by a factor of 2.
Reflect the graph across the x-axis.
Shift the graph 2 units down.

2. Find **(a)** $f + g$, **(b)** $f - g$, **(c)** fg, and **(d)** f/g, and state their domains.

(i) $f(x) = 2^x, \qquad g(x) = 3^x$

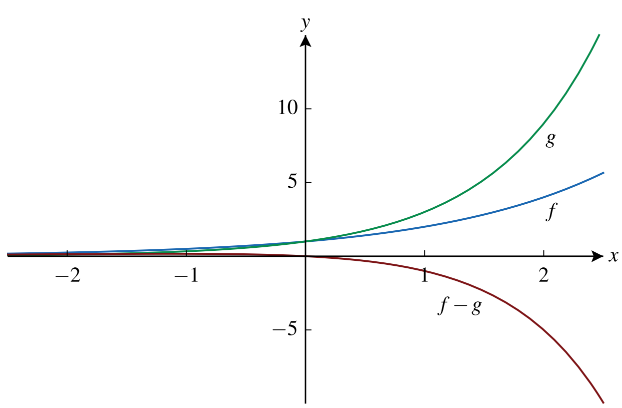
$$(f + g)(x) = f(x) + g(x)$$

$$= 2^x + 3^x \qquad \text{Domain: } \mathbb{R}$$



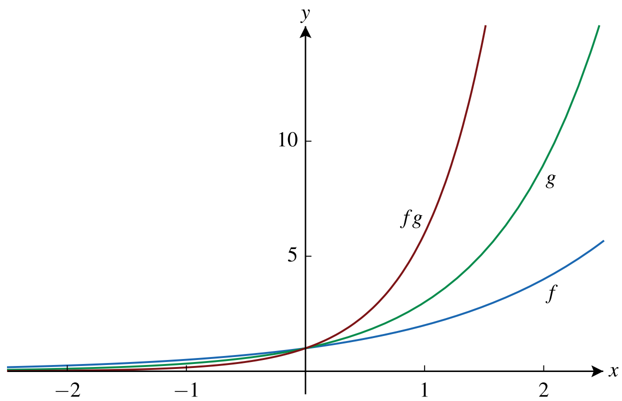
$$(f - g)(x) = f(x) - g(x)$$

$$= 2^x - 3^x \qquad \text{Domain: } \mathbb{R}$$



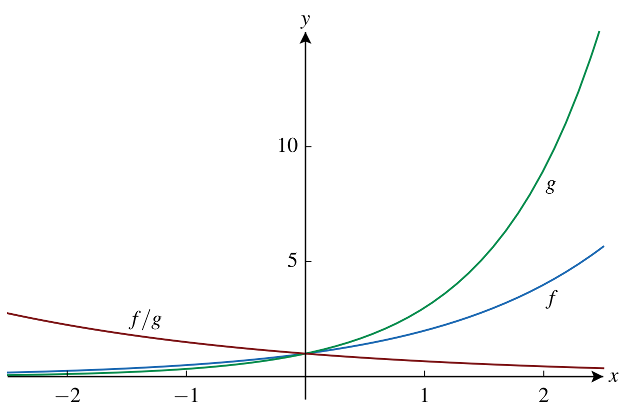
$$(fg)(x) = f(x) \cdot g(x)$$

$$= 2^x \cdot 3^x = 6^x \qquad \text{Domain: } \mathbb{R}$$



$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

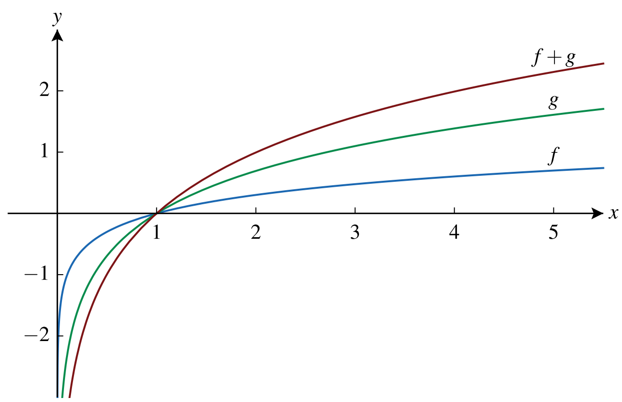
$$= \frac{2^x}{3^x} = \left(\frac{2}{3}\right)^x \qquad \text{Domain: } \mathbb{R}$$



(ii) $f(x) = \log x, \qquad g(x) = \ln x$

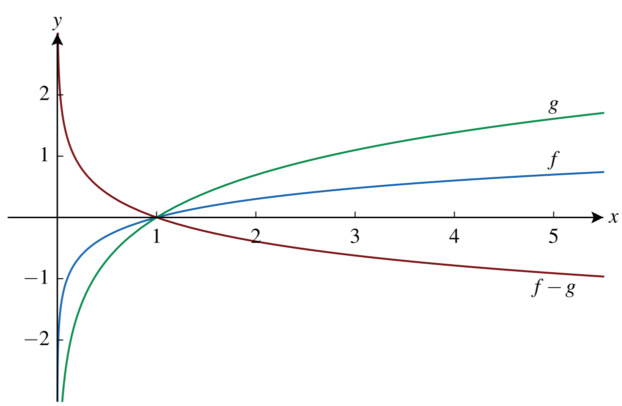
$$(f + g)(x) = f(x) + g(x)$$

$$= \log x + \ln x \qquad \text{Domain: } x > 0$$



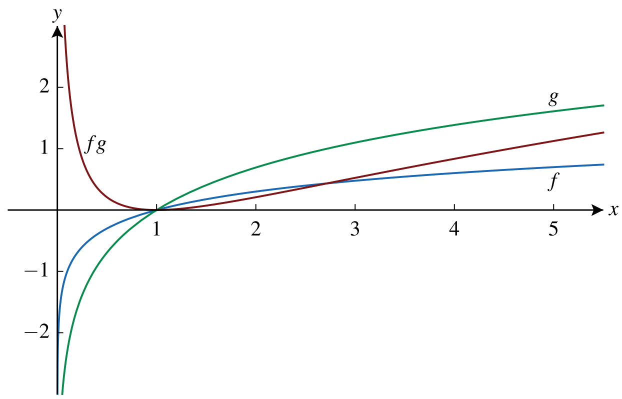
$$(f - g)(x) = f(x) - g(x)$$

$$= \log x - \ln x \qquad \text{Domain: } x > 0$$



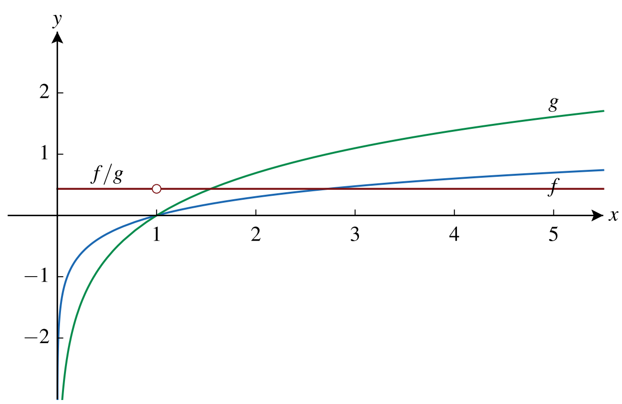
$$(fg)(x) = f(x) \cdot g(x)$$

$$= (\log x) \cdot (\ln x) \qquad \text{Domain: } x > 0$$



$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

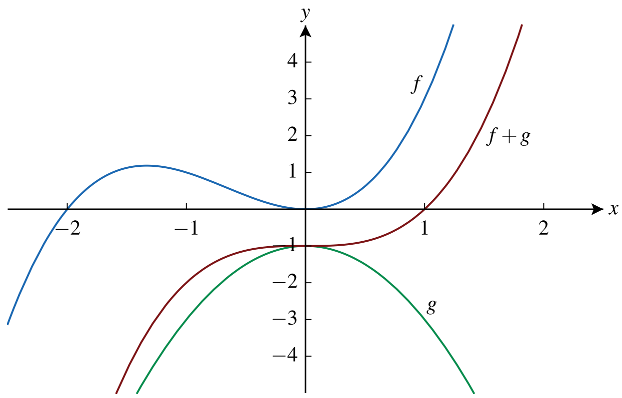
$$= \frac{\log x}{\ln x} = \qquad \text{Domain: } x > 0, \quad x \neq 1$$



(iii) $f(x) = x^3 + 2x^2, \qquad g(x) = -2x^2 - 1$

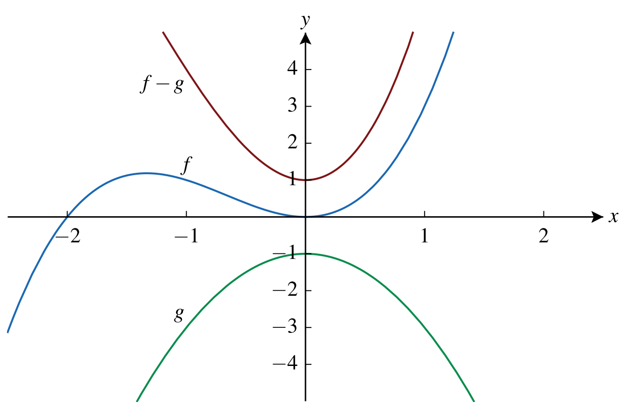
$$(f + g)(x) = f(x) + g(x)$$

$$= x^3 + 2x^2 + (-2x^2 - 1) = x^3 - 1 \qquad \text{Domain: } \mathbb{R}$$



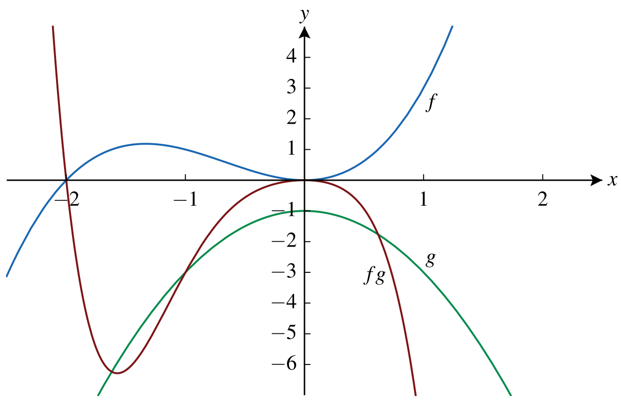
$$(f - g)(x) = f(x) - g(x)$$

$$= (x^3 + 2x^2) - (-2x^2 - 1) = x^3 + 4x^2 + 1 \qquad \text{Domain: } \mathbb{R}$$



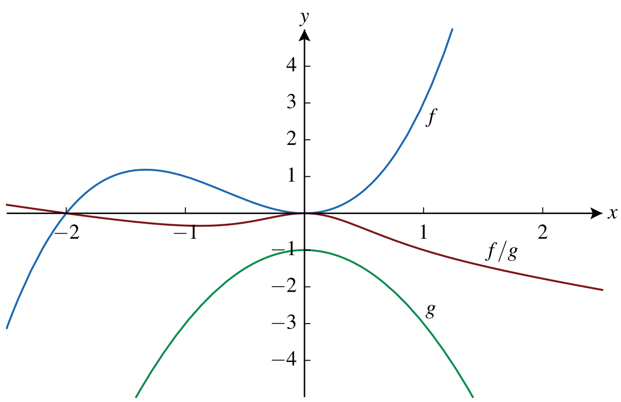
$$(fg)(x) = f(x) \cdot g(x)$$

$$= (x^3 + 2x^2)(-2x^2 - 1) = -2x^5 - 4x^4 - x^3 - 2x^2 \qquad \text{Domain: } \mathbb{R}$$



$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$= \frac{x^3 + 2x^2}{-2x^2 - 1} = \frac{x^2(x + 2)}{-(2x^2 + 1)} \qquad \text{Domain: } \mathbb{R}$$

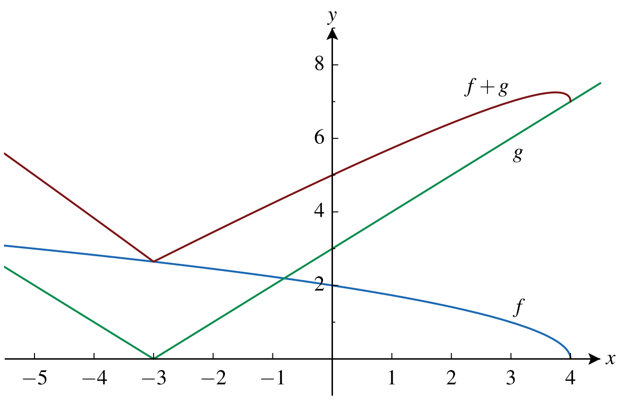


(iv) $f(x) = \sqrt{4-x}, \quad g(x) = |x+3|$

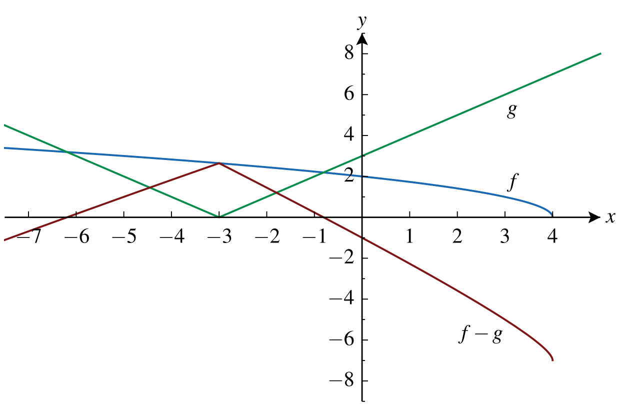
$f(x) = \sqrt{4-x}; \qquad 4 - x \geq 0 \;\Rightarrow\; -x \geq -4 \;\Rightarrow\; x \leq 4 \qquad$ Domain: $x \leq 4$

$g(x) = |x+3| \qquad$ Domain : $\mathbb{R}$

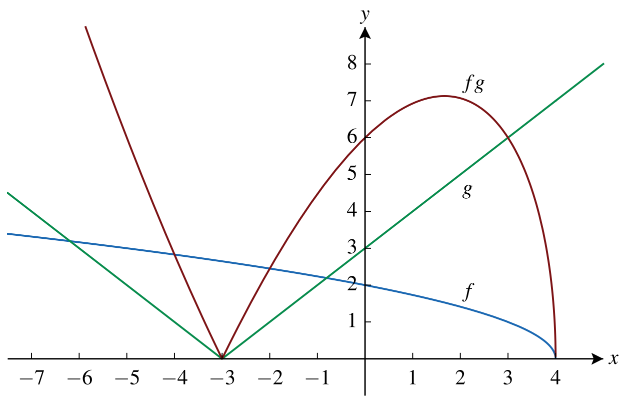
$(f+g)(x) = f(x) + g(x) = \sqrt{4-x} + |x+3| \qquad$ Domain: $x \leq 4$



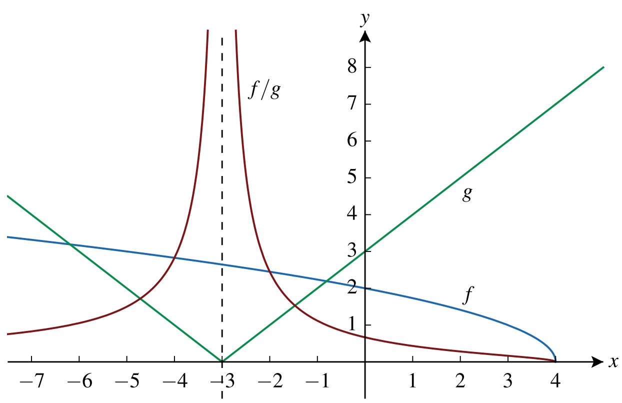
$(f-g)(x) = f(x) - g(x) = \sqrt{4-x} - |x+3| \qquad$ Domain: $x \leq 4$



$(fg)(x) = f(x) \cdot g(x) = \sqrt{4-x} \cdot |x+3|$ Domain: $x \leq 4$



$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{4-x}}{|x+3|}$ Domain: $x \leq 4,\ x \neq -3$

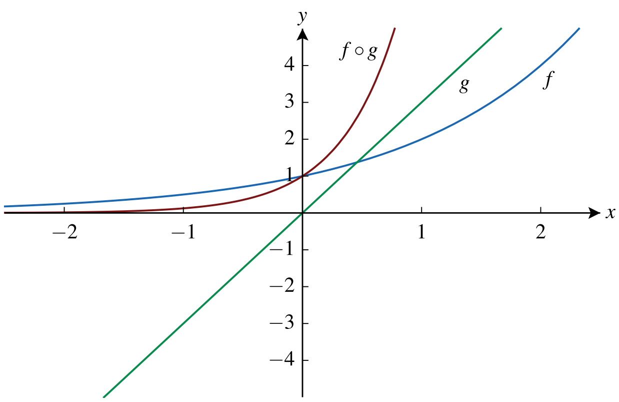


3. Find the functions **(a)** $f \circ g$, **(b)** $g \circ f$, **(c)** $f \circ f$, and **(d)** $g \circ g$, and state their domains.

(i) $f(x) = 2^x, \qquad g(x) = 3x$

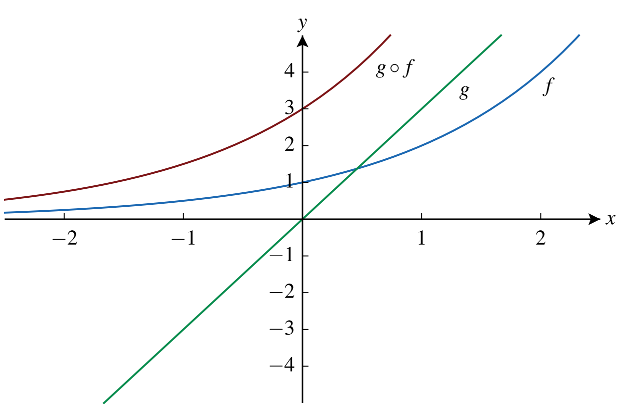
$$(f \circ g)(x) = f(g(x)) = f(3x) = 2^{3x} = 8 \cdot 2^x$$

Domain: $x \in \mathbb{R}$



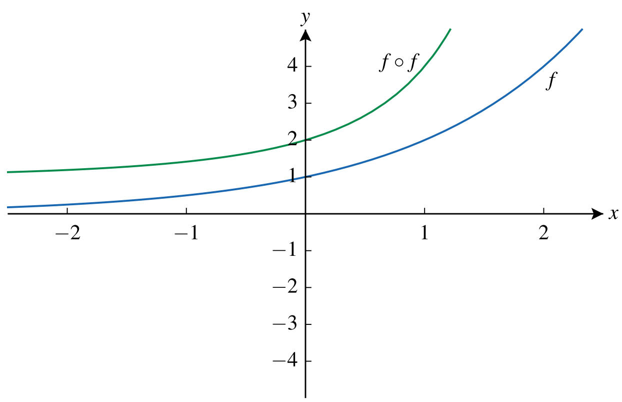
$$(g \circ f)(x) = g(2^x) = 3 \cdot 2^x$$

Domain: $x \in \mathbb{R}$



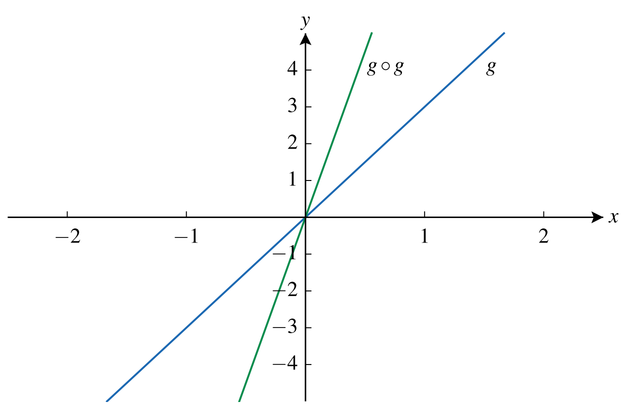
$(f \circ f)(x) = f(2^x) = 2^{(2^x)}$

Domain: $x \in \mathbb{R}$



$(g \circ g)(x) = g(3x) = 3(3x) = 9x$

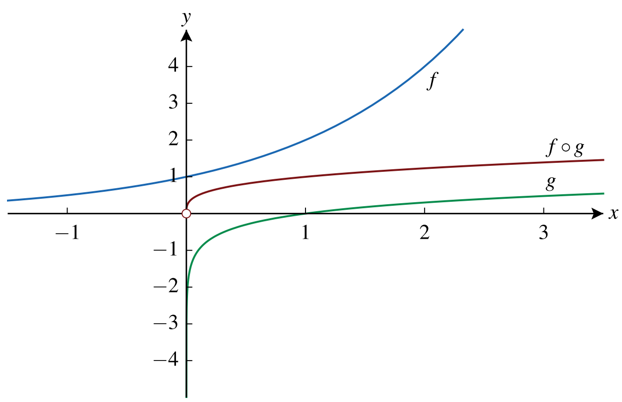
Domain: $x \in \mathbb{R}$



(ii) $f(x) = 2^x, \qquad g(x) = \log x$

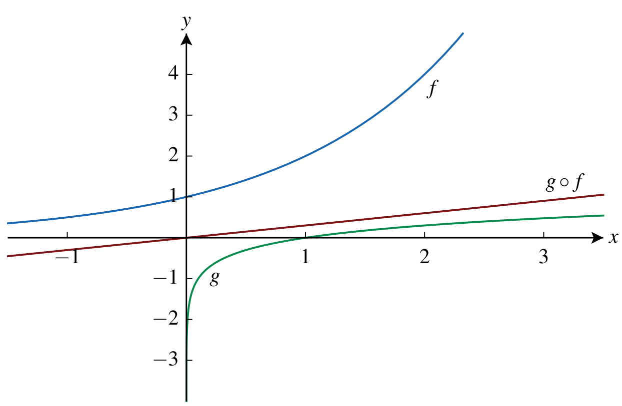
$$(f \circ g)(x) = f(g(x)) = f(\log x) = 2^{\log x}$$

Domain: $x > 0$



$$(g \circ f)(x) = g(2^x) = \log(2^x) = x \log 2$$

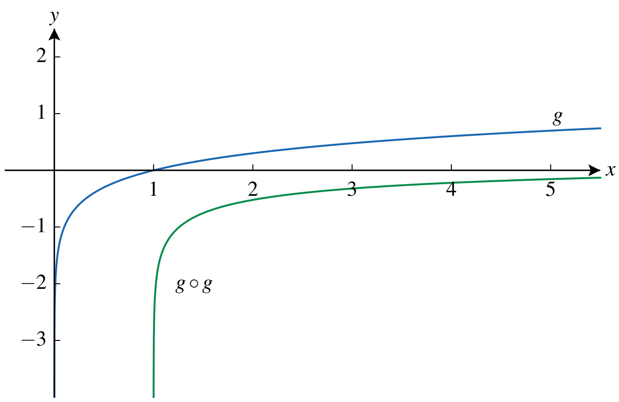
Domain: $x \in \mathbb{R}$



$$(f \circ f)(x) = f(2^x) = 2^{(2^x)}$$

$$(g \circ g)(x) = g(\log x) = \log(\log x)$$

Domain: $\log x > 0 \quad \Rightarrow \quad x > 1$

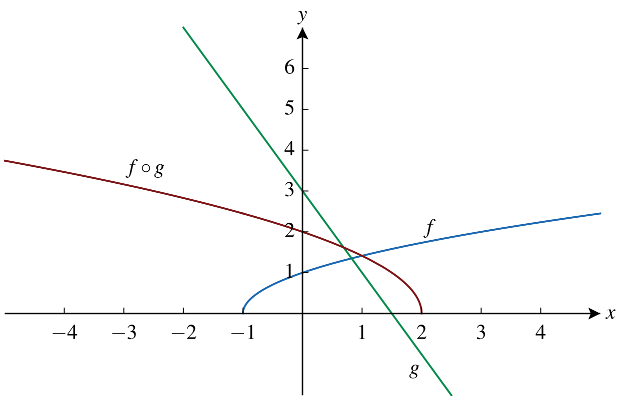


(iii) $f(x) = \sqrt{x+1}, \quad g(x) = -2x+3$

$f(x) = \sqrt{x+1}; \qquad x+1 \geq 0 \;\Rightarrow\; x \geq -1$

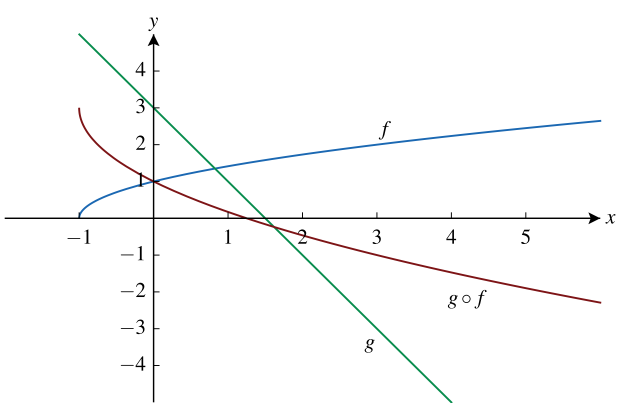
$(f \circ g)(x) = f(g(x)) = f(-2x+3) = \sqrt{(-2x+3)+1} = \sqrt{-2x+4}$

Domain: $-2x+4 \geq 0 \;\Rightarrow\; -x+2 \geq 0 \;\Rightarrow\; -x \geq -2 \;\Rightarrow\; x \leq 2$



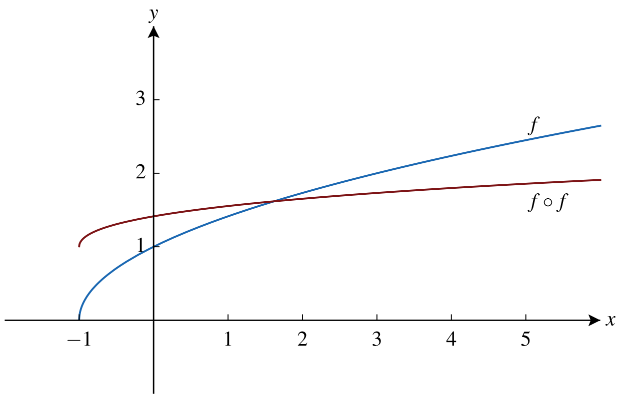
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x+1}) = -2\sqrt{x+1}+3$

Domain: $x \geq -1$

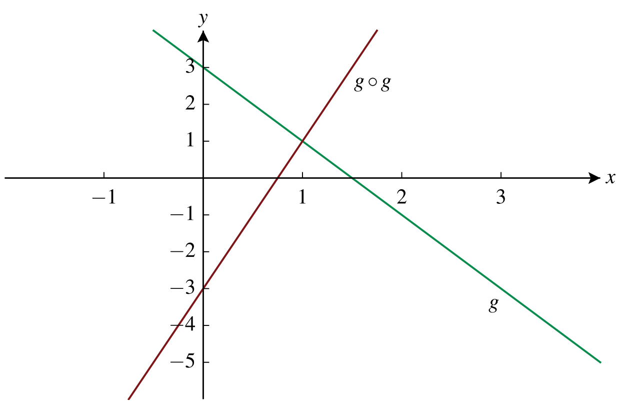


$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x+1}) = \sqrt{\sqrt{x+1}+1}$$

$$\sqrt{x+1}+1 \geq 0 \quad \Rightarrow \quad \sqrt{x+1} \geq -1 \quad \Rightarrow \quad \text{Domain: } x \geq -1$$



$$(g \circ g)(x) = g(g(x)) = g(-2x+3) = -2(-2x+3)+3 = 4x-3 \qquad \text{Domain: } \mathbb{R}$$



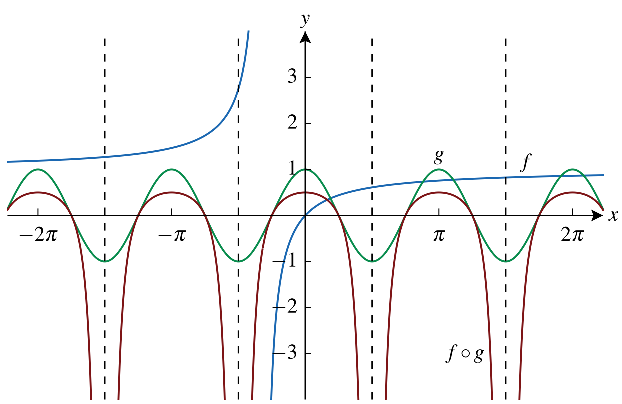
(iv) $f(x) = \dfrac{x}{x+1}, \quad g(x) = \cos 2x$

Domain of f: $\mathbb{R},\ x \neq -1$ Domain of g: $\mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(\cos 2x) = \frac{\cos 2x}{\cos 2x + 1}$$

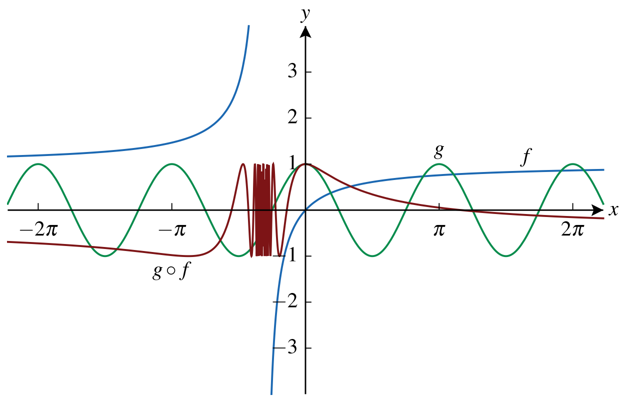
$$\cos 2x + 1 = 0 \ \Rightarrow\ \cos 2x = -1 \ \Rightarrow\ 2x = \pi + 2n\pi \ \Rightarrow\ x = \frac{\pi}{2} + n\pi$$

Domain: $\mathbb{R},\ \ x \neq \dfrac{\pi}{2} + n\pi,\ \ n = 0, \pm 1, \pm 2, \ldots$



$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{x+1}\right) = \cos\left(2\left(\frac{x}{x+1}\right)\right)$$

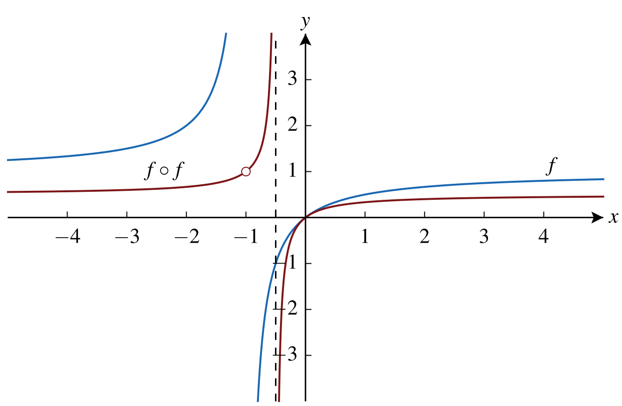
Domain: $\mathbb{R}, \;\; x \neq -1$



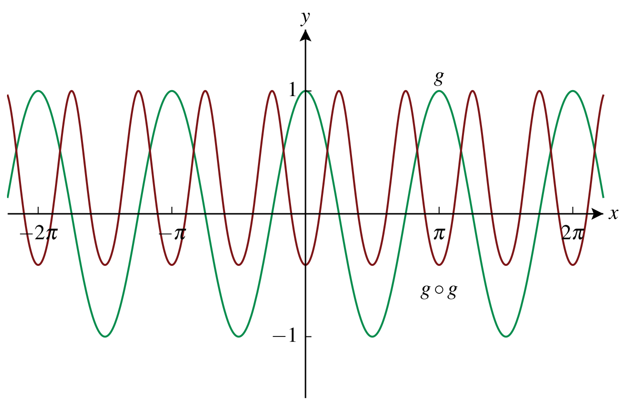
$$(f \circ f)(x) = f(f(x)) = f\left(\frac{x}{x+1}\right) = \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1}$$

$$= \frac{\frac{x}{x+1}}{\frac{x+x+1}{x+1}} = \frac{x}{x+1} \cdot \frac{x+1}{2x+1} = \frac{x}{2x+1}$$

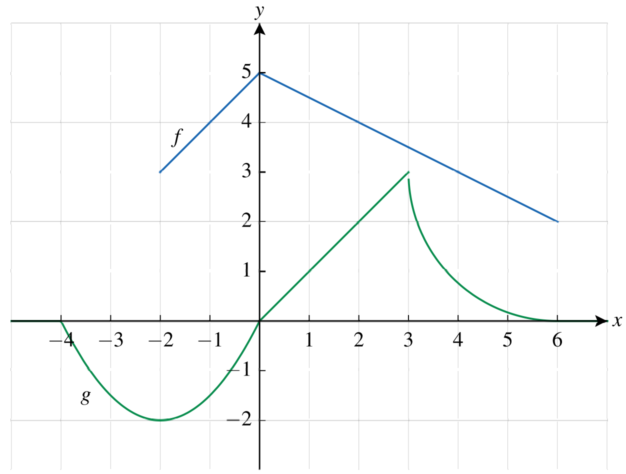
Domain: $\mathbb{R}, \;\; x \neq -1, \; -\frac{1}{2}$



$(g \circ g)(x) = g(g(x)) = g(\cos 2x) = \cos(2\cos 2x)$
Domain: $\mathbb{R}$



4. Use the graphs of f and g to evaluate each expression, or explain why it is undefined.



(a) $f(g(2)) = f(2) = 4$

(b) $g(f(4)) = g(3) = 3$

(c) $(f \circ g)(-2) = f(g(-2)) = f(-2) = 3$

(d) $(g \circ f)(6) = g(f(6)) = g(2) = 2$

(e) $(g \circ g)(-2) = g(g(-2)) = g(-2) = -2$

(f) $(f \circ f)(0) = f(f(0)) = f(5) = \dfrac{5}{2}$

(g) $(g \circ f)(5) = g(f(5)) = g\left(\dfrac{5}{2}\right) = \dfrac{5}{2}$

(h) $(f \circ g \circ f)(4) = f(g(f(4))) = f(g(3)) = f(3) = \dfrac{7}{2}$

(i) $(g \circ f \circ g)(-2) = g(f(g(-2))) = g(f(-2)) = g(3) = 3$

5. Find a formula for the inverse of the function.

(a) $f(x) = 1 + \sqrt{3 + 7x}$

$$y = 1 + \sqrt{3 + 7x}$$

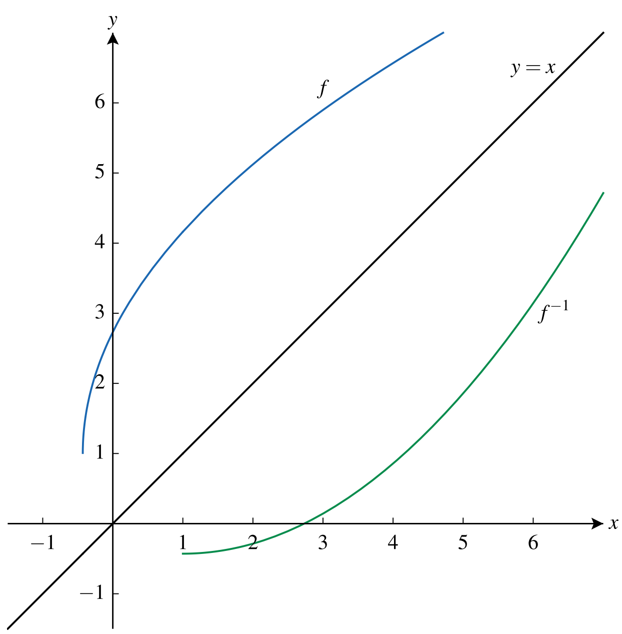
$$y - 1 = \sqrt{3 + 7x}$$

$$(y - 1)^2 = 3 + 7x$$

$$(y - 1)^2 - 3 = 7x$$

$$\frac{1}{7}\left((y - 1)^2 - 3\right) = x$$

$$f^{-1}(x) = \frac{1}{7}\left((x - 1)^2 - 3\right)$$



(b) $f(x) = \dfrac{4x-1}{2x+3}$

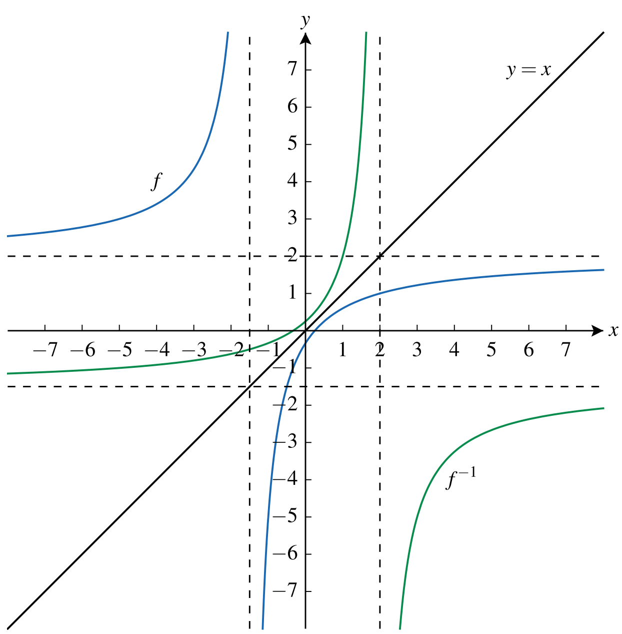
$$y = \frac{4x-1}{2x+3}$$

$$y(2x+3) = 4x-1 \quad \Rightarrow \quad 2xy + 3y = 4x - 1$$

$$2xy - 4x = -3y - 1 \quad \Rightarrow \quad x(2y-4) = -(3y+1)$$

$$x = \frac{-(3y+1)}{2y-4}$$

$$f^{-1}(x) = \frac{-(3x+1)}{2x-4}$$



(c) $f(x) = \sqrt{1 - x^2}, \quad 0 \leq x \leq 1$

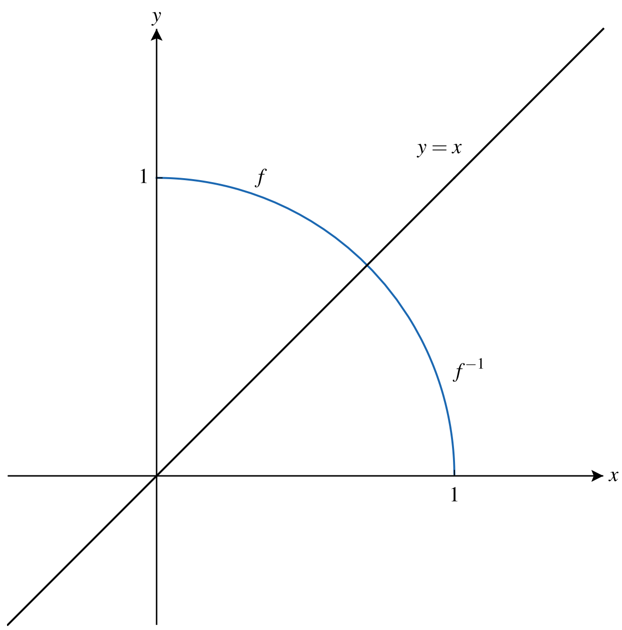
$$y = \sqrt{1 - x^2}$$

$$y^2 = 1 - x^2$$

$$x^2 = 1 - y^2$$

$$x = \sqrt{1 - y^2}$$

$$f^{-1}(x) = \sqrt{1 - x^2}$$



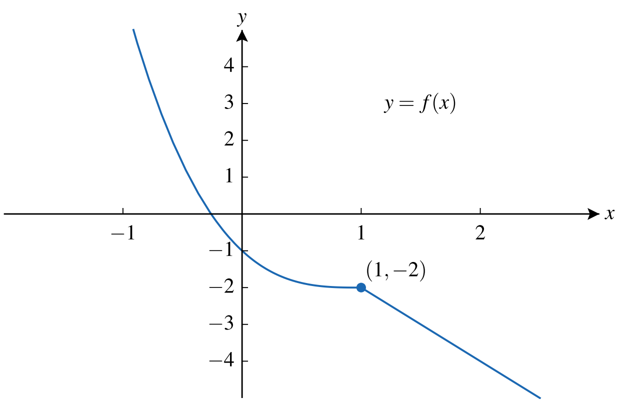
Overtime Problems

1. The function f is defined by

$$f(x) = \begin{cases} |x-1|^3 - 2 & \text{if } x < 1 \\ -2x & \text{if } x \geq 1 \end{cases}$$

Find the intervals on which the function is decreasing; increasing.

$$f(x) = \begin{cases} |x-1|^3 - 2 & \text{if } x < 1 \\ -2x & \text{if } x \geq 1 \end{cases} = \begin{cases} (-(x-1))^3 - 2 & \text{if } x < 1 \\ -2x & \text{if } x \geq 1 \end{cases} = \begin{cases} -(x-1)^3 - 2 & \text{if } x < 1 \\ -2x & \text{if } x \geq 1 \end{cases}$$



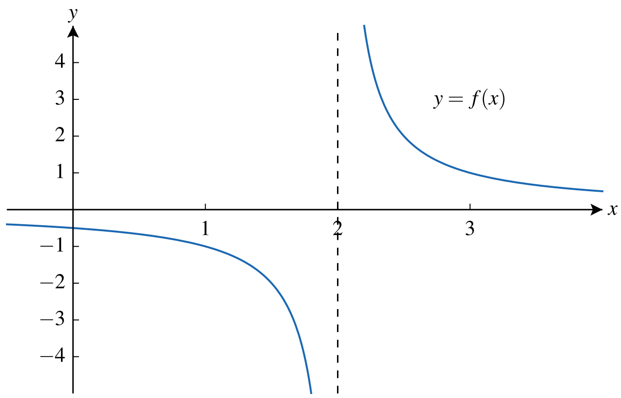
The function f is decreasing on $(-\infty, \infty)$.

2. Rational functions: vertical asymptotes and holes. See TNPC, 9/28

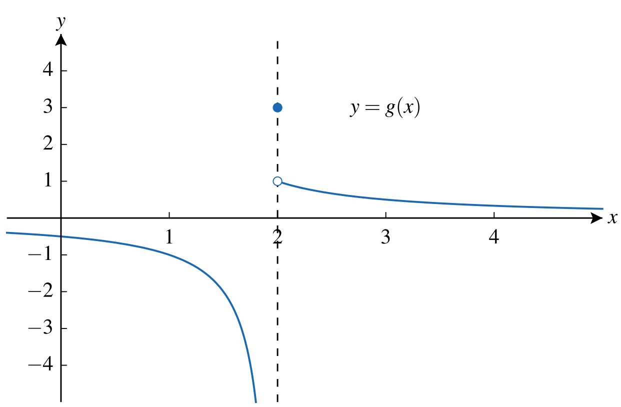
Given a rational function:

Holes occur when factors from the numerator and the denominator cancel. When a factor in the denominator does not cancel, it produces a vertical asymptote. Both holes and vertical asymptotes restrict the domain of a rational function.

Consider $f(x) = \dfrac{(x-2)}{(x-2)^2}$



$$g(x) = \begin{cases} \dfrac{1}{x-2} & \text{if } x < 2 \\ \dfrac{(x-2)}{(x-2)(x-1)} & \text{if } x > 2 \\ 3 & \text{if } x = 2 \end{cases}$$



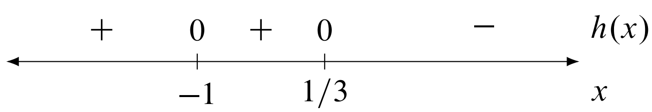
3. The function f is defined by $f(x) = -2(x+3)^2(x-1)$. The function h is defined by $h(x) = 2f(3x)$.

Find the intervals on which $h(x) \geq 0$.

$$h(x) = 2f(3x)$$

$$= 2\left[-2(3x+3)^2(3x-1)\right]$$

$$= -4 \cdot 3^2(x+1)^2(3x-1) = -36(x+1)^2(3x-1)$$



$h(x) \geq 0$ on the interval $\left(-\infty, \frac{1}{3}\right]$