FROM NUMBERS TO SYMBOLS TO GRAPHS WITH A TI-92 PLUS

Armando M. Martínez-Cruz
California State University, Fullerton
amartinez-cruz@fullerton.edu

José N. Contreras
The University of Southern Mississippi
Jose.Contreras@usm.edu

Presented at

2001 T^3
Teachers Teaching with Technology
International Conference

Columbus, Ohio
March 18, 2001

INTRODUCTION

A central goal of current reform in mathematics education is for students to become mathematical problem solvers (e.g., NCTM, 1989, 2000). In this task, teachers play an essential role. Teachers are expected to model and emphasize "aspects of problem solving, including formulating and posing problems, solving problems using different strategies, verifying and interpreting results, and generalizing solutions," (NCTM, 1991, p. 95). How can teachers achieve these goals? Technology can, in particular, support to achieve this goal.

Technology has been identified as a particular feature of high-quality education in current reform mathematics education documents. For instance, The NCTM's *Principles and Standards for School Mathematics* (NCTM, 2001) describes technology as "essential in teaching and learning mathematics; [for] it influences the mathematics that is taught and enhances students' learning" (p. 11). In the same vein, the document states that

> electronic technologies…are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately. They can support investigation by students in every area of mathematics, including geometry, statistics, algebra, measurement, and number. When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving (p. 24).

Three critical issues in the technology principle in *Principles and Standards for School Mathematics* (NCTM, 2001): Technology enhances mathematics learning, supports effective mathematics teaching, and influences what mathematics is taught.

Technology enhances mathematics learning
"Technology can help students learn mathematics. For instance, with calculators and computers students can examine more examples or representational forms than are feasible by hand, so they can make and explore conjectures easily" (NCTM, 20001, p. 24).

Technology supports effective mathematics teaching
"Teachers should use technology to enhance their students' learning opportunities by selecting or creating mathematical tasks that take advantage of what technology can do efficiently and well—graphing, visualizing, and computing" (NCTM, 2001, p. 25-26).

Technology influences what mathematics is taught
"Technology not only influences how mathematics is taught and learned but also affects what is taught and when a topic appears in the curriculum. …The study of algebra need not be limited to simple situations in which symbolic manipulation is relatively straightforward. Using technological tools, students can reason about more-general issues, such as parameter changes, and they can model and solve complex problems that were heretofore inaccessible to them. Technology also blurs some of the artificial separations among topics in algebra, geometry, and data analysis by allowing students to use ideas from one area of mathematics to better understand another area of mathematics" (NCTM, 2001, p. 26-27).

Problem posing
Problem formulation or problem posing is increasingly receiving attention from both a curricular and pedagogical perspective (NCTM, 2000). For instance, "students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking," (NCTM, 2000, p. 52).

"Problem posing" applies in general to three distinct forms of mathematical cognitive activity (Silver, 1994):

(a) a presolution posing, in which one generates original problems from a presented stimulus situation;
(b) within-solution posing, in which one reformulates a problem as it is being solved; and
(c) postsolution posing, in which one modifies the goals or conditions of an already solved problem to generate new problems.

This presentation illustrates our efforts to combine problem solving, problem posing and technology use. We investigate a problem, from which we took the goal again to make it into an open problem, with the TI-92 plus. The investigation includes numerical, graphic and symbolic approaches. Then we propose two related problems, one is a particularization and the other one is a generalization of the original problem. We have investigated this problem with prospective and inservice secondary mathematics teachers, and with high school students at different levels. One student called the problem, the "never-ending problem" due to the many aspects that emerged in the investigations.

The problem that we investigate is the following:

What can be said about the product of four consecutive integers?

However this is not the way the problem is stated in the original source (Shklarsky, Chentzov, & Yaglom, 1993). We simply took away the goal away. A full discussion of this approach appears somewhere else (Martínez-Cruz, & Contreras, submitted).

Students can observe that the product is divisible by 2, 3 and 4, and sometimes by 5. With a little help, some may recognize and prove that the product is divisible by 6 and by 1. Figure 1 shows the quotient formed by the product of four consecutive integers and 6, 8, 12 and 24, respectively. Notice that we can also use negative numbers. Here we had to encourage students to do so, for none of them thought of it. To prove that the product is divisible by 6, 12 and 24 we use the following result.

Divisibility by Product Theorem (DPT). If $a|p$ and $b|p$ and GCF(a, b) = 1 with a, b and p positive integers then $ab|p$. (The result is easily generalized to a, b, and p negative integers.)

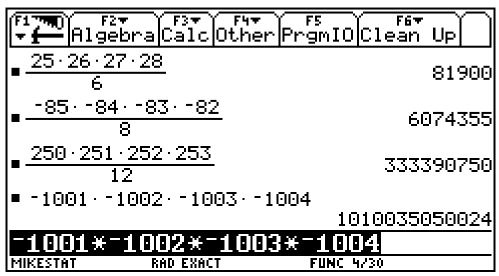

Fig. 1

Some of our students said that the product is divisible by 24 and by 48 and that the numbers ended in 0 or in 4. Actually they discovered that the numbers that ended in 4, also ended in 04. Furthermore, one was adventurous to claim that those numbers ended in 024. Table 1 shows some instances.

Product of four consecutive integers, none a multiple of five.	First integer
-11*-12*-13*-14 = 24024	-11
1*2*3*4 = 24	1
6*7*8*9 = 3024	6
11*12*13*14 = 24,024	11
16*17*18*19 = 93,024	16
21*22*23*24 = 255,024	21

Table 1

We will prove that the product minus 24 is a multiple of 1000. This proof involves first realizing that the smallest number of the four consecutive integers (none of which is a multiple of 5) has the form $5n + 1$ (with n an integer, see second column of table 2), and then substituting this form into the product minus 24 (figure 2). Finally, write every integer n either as $4k$, $4k + 1$, $4k + 2$ or

$4k + 3$ (figure 14), substitute it into the product and factor it. Substitution and factoring (algebra menu) are combined in the TI-92 in Figure 3. This proves the result.

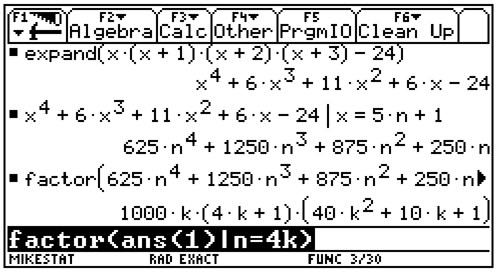

Fig. 2

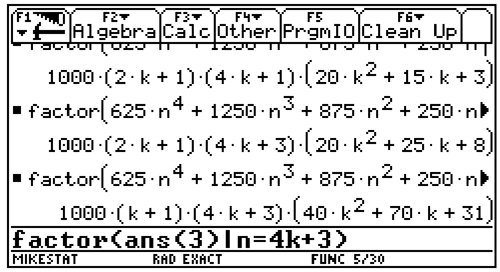

Fig. 3

Notice that that all the products (with no multiple of five) minus 24 are multiples of 3000. It is easy to verify that the product of four consecutive integers minus 24 is also divisible by 3 (If a and b are divisible by p then $a + b$ is divisible by p.) Since we just proved that the product of four consecutive integers minus 24 is divisible by 1000, we can apply DPT (since GCF(3,1000) = 1). This concludes the proof.

What if we added 1 to the product???

See figure 4 for some examples. Some students recognize them as perfect square. Figure 4 also shows the result with “large numbers” and the algebraic proof of the result.

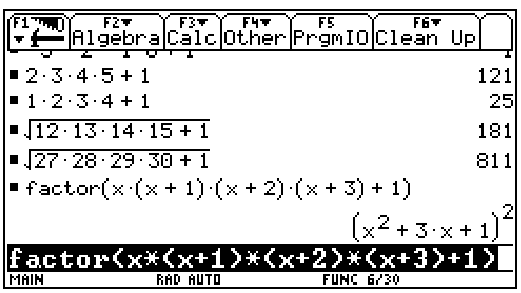

Fig. 4

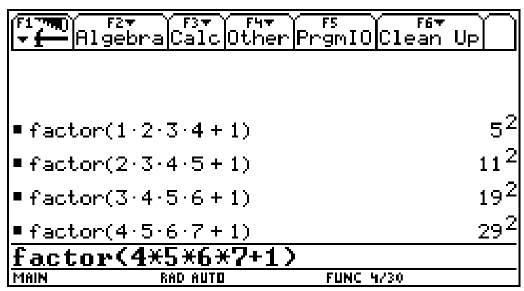

Fig. 5

A different approach using multiplication

Here is a different approach. In figure 5 we use the calculator to factor the products plus one. In this case, we asked the students to predict the number that we have to square in order to get the product plus one, by using only the four numbers given.

Some students observed that the product plus 1 is the same if we multiply the middle numbers, subtract one and then square the answer. Other students discovered that we could multiply the outside numbers, add one and then square the answer. Figure 6 shows an instance of both situations and the generalization using algebra. This is the same result given in figure 4.

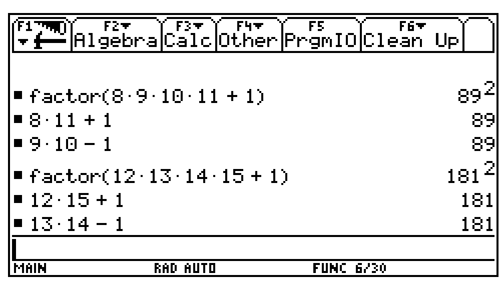

Fig. 6

Hence we have used the calculator to discover numerical patterns and provide the solution. Factoring is a good tool but if you do not have a symbolic manipulator it may be hard to use. Pattern recognition is a useful technique that did not require factoring. Now we use a graphing approach.

A conjecture from graphing and why do we have to add 1???

Figure 7 shows the graph of the function $f(x) = x*(x+1)*(x+2)*(x+3)$. This graph suggests that if we shift the graph up "a little bit" we might have a positive function with two double roots. But how much do we have to shift it up? Of course by the minimum value!!

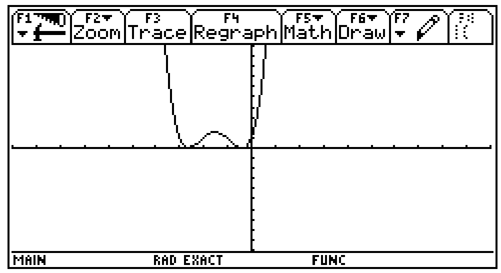

Fig. 7

In figure 8 we defined a function that is the product of four consecutive integers. The figure also shows the derivative of this function and its roots, which we obtained by solving the equation "the derivative function equals to zero." The last step is to evaluate the function at the critical values to find the maxima or minima. In this case, -1 is a minimum value. Hence we have to shift the graph one unit up.

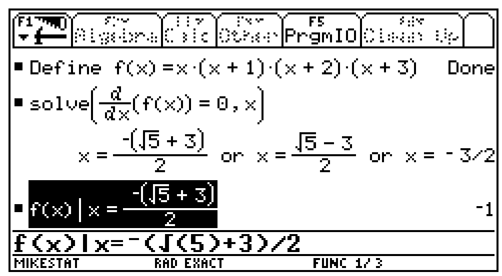

Fig. 8

Extensions of the problem

First, what happens if we had less or more consecutive integers??? One of our students proposed to explore the case when we had three consecutive integers. He also discovered the following result: Add the middle number to the product and you get a perfect cube. Figure 9 shows some instances.

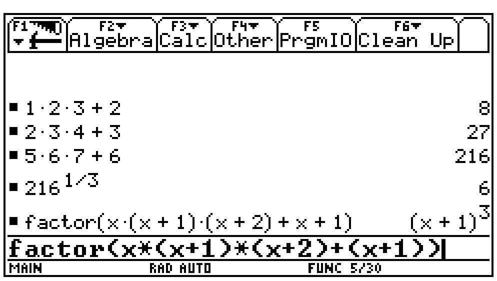

Fig. 9

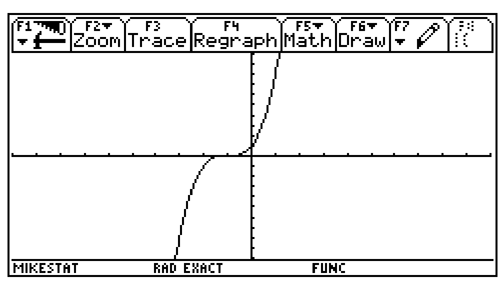

Fig. 10

Figure 10 shows the graph of $f(x) = x*(x+1)*(x+2)+(x+1)$. The graph suggests that we have a cubic translated to the left. The last row in figure 9 confirms this conjecture.

Our second generalization of this problem involves the product of four integers with common difference 2, for instance {10, 12, 14, 16}. What can be said about the product of these numbers?? Here instead of looking at the numbers, we look at the graph of $f(x) = x*(x+2)*(x+4)*(x+6)$ (figure 11) and proceed as we did before. First, we notice that we need to shift the graph up to make it positive. This shift is given by the minimum value, which we find using the Math menu.

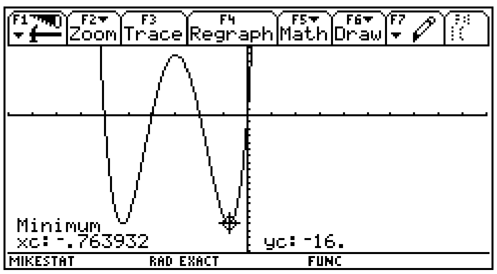

Fig. 11

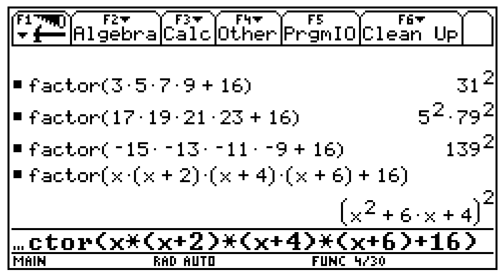

Fig. 12

So our conjecture is that the product of four consecutive integers with common difference 2, plus 16 is a perfect square. Figure 12 shows some instances and the algebraic proof of this result.

A final generalization

Take four consecutive integers with common difference *k*. Can we add a value to their product so that it results in a perfect square???? The answer is "yes."

References

Martínez-Cruz, A. M. & Contreras, J. (submitted). What if we take the goal away from some problems? An adventure in problem solving and problem posing with technology.

NCTM. (2000). *Principles and standards for school mathematics.* Reston, VA: The Author.

Shklarsky, D. O., Chentzov, N. N. & Yaglom, I. M., (1993). *The USSR Olympiad problem book. Selected problems and theorems of elementary mathematics*, p. 21. New York: Dover.