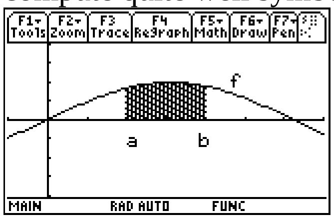
Discovering A Fundamental Fact about Integration

by Dave Slomer

You have investigated Riemann sums and have a good feel for what $\int_a^b f(x)dx$ means [the area between *f* and the *x*-axis from *a* to *b*], what it looks like [the graph below], and how to approximate it [by summing areas of a lot of approximating rectangles]. But there is a ***symbolic*** way to calculate many such integrals, giving <u>exact</u> results. Luckily, the TI-89 can compute quite well symbolically.



Exercise 1: Use your head, a sketch on coordinate axes, and a well-known formula from geometry (but ***not*** your TI-89) to find the <u>exact</u> value of $\int_0^1 x dx$. (The only calculus involved is in correctly interpreting the symbols to make the corresponding sketch.)

Exercise 2: To find some more exact values of integrals, you can quit using your head (well, not <u>entirely</u>!) and let your '89 do the work. Your '89 can even perform symbolic operations in its Data/Matrix Editor, by having one column refer to another.

- Go into your TI-89's **Data/Matrix Editor** (press [APPS] [6] [3] ▼ ▼ and type a **Data** variable name that is not in use, such as **integdat**, and press [ENTER]).
- Move the cursor to row 1 of column **c1** and type **x^2** [ENTER].
- Move the cursor so that **c2** is highlighted (press ► ▲ ▲) and type **∫ (c1,x,0,1)**[ENTER].
  [The ∫ symbol is [2nd] [7].]

The screen should look like figure 1 if you move the cursor so that **c2** is highlighted.

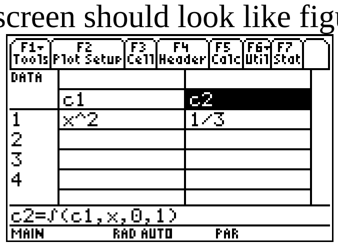


Fig. 1

Because of the formula stored in **c2**, the screen says that $\int_0^1 x^2 dx = \frac{1}{3}$, since row 1 of column **c1** contains **x^2** and row 1 of column **c2** contains 1/3. Overall, the formula in **c2** tells the '89 to evaluate the integral from 0 to 1 for <u>any</u> function of *x* in column **c1** and to put the result in column **c2**. So, column **c2** contains the value of $\int_0^1 c1(x)dx$ for each function of *x* in column **c1**.

In rows 2 through 4 of column **c1**, type **x^3** [ENTER], **x^4** [ENTER], and **x^5** [ENTER]. The screen will look like the one in figure 2, but the '89 will have calculated 3 more integrals in column **c2**.

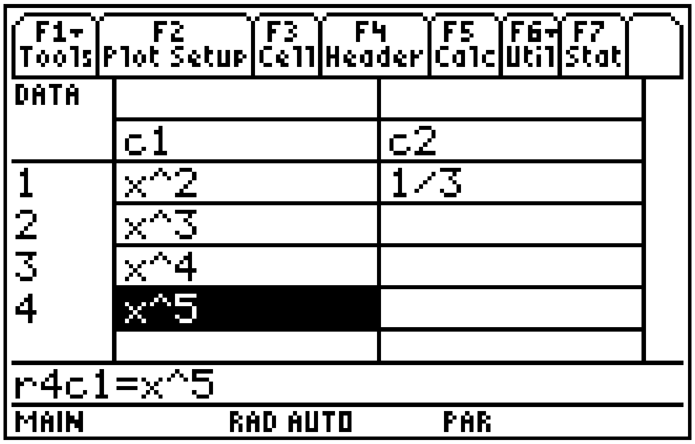


Fig. 2

Fill in column **c2** of the data grid above with the values of the integrals returned by your '89 and then use *inductive reasoning* to try to <u>guess</u> a formula: $\int_0^1 x^n\,dx =$ _____.

Exercise 3: Apply your Exercise 2 formula to the <u>triangle</u> whose area you found in Exercise 1. Did you get the same result as in Exercise 1? If not, re-think Exercise 1 or 2.

Exercise 4: Using your head, a sketch, and a well-known formula from geometry (but ***<u>not</u>*** your '89), complete the following formula: $\int_0^b x dx \;=\;$ _____.

Exercise 5: In the Data/Matrix Editor [press $\boxed{\text{APPS}}$ $\boxed{6}$ $\boxed{\text{ENTER}}$ to re-use the **current** data grid from Exercise 2], modify the formula stored in **c2** to find a more general result: $\int_0^b c1(x)dx$ [move the cursor so that **c2** is highlighted, press $\boxed{\text{ENTER}}$ and change the 1 to a **b** so that the command line reads · **(c1,x,0,b)**, and press $\boxed{\text{ENTER}}$]. The screen should look like the one in figure 3 if you move the cursor so that **c2** is highlighted.

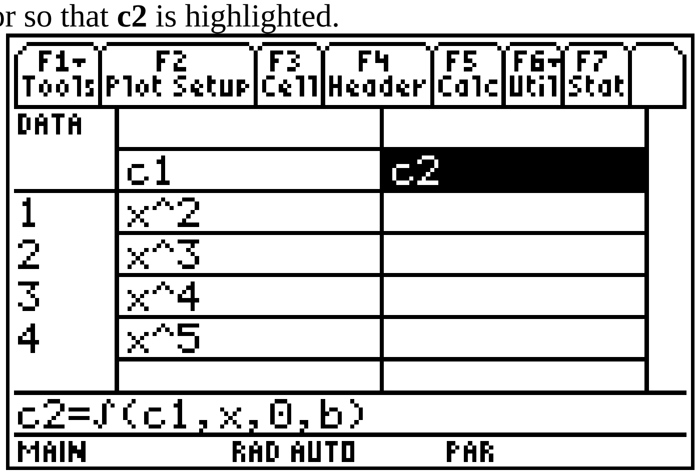


Fig. 3

Fill in column **c2** of the data grid above with the values of the integrals returned by your '89 and then use inductive reasoning to make a ***<u>guess</u>*** at a formula: $\int_0^b x^n\,dx =$ _____.

Exercise 6: Apply your Exercise 5 formula to the <u>triangle</u> whose area you found in Exercise 4. Did you get the same result both ways? If not, re-think Exercise 4 or 5.

Exercise 7: Change column **c1** of the data grid to read as in figure 4. [To do so, move the cursor to row 1, column **c2**, type **cos(x)** {or press [2nd] **Z**}, and press [ENTER], which puts you in row 2. Repeat the process for rows 2 through 4 {**sin(x)** is [2nd] **Y**; **e^x** is [◆] **X**}.]

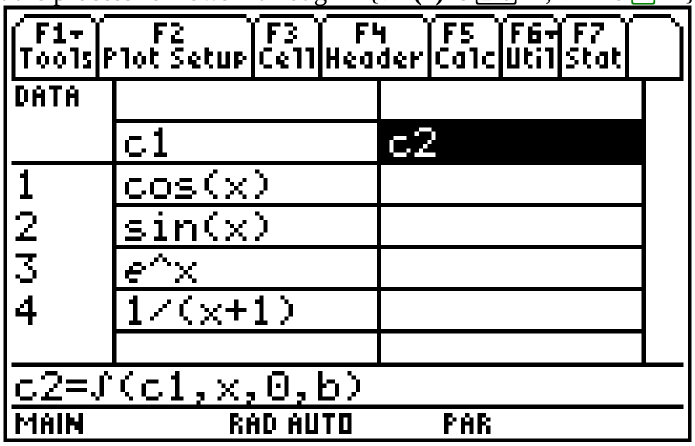


Fig. 4

Fill in column **c2** of the data grid above with the values returned by your ’89. Then apply inductive reasoning to these results and the results from Exercise 2 to make a <u>guess</u> at a **more general formula**: $\int_0^b f(x)dx =$ _____ *Hint: think derivatives and be prepared to write a lot more than will fit into the small blank above! But don’t dwell on it. Move on if you can’t get it within a reasonable time. Exercise 9 may help.*

Exercise 8: Assume that $a < b$. Sketch the regions represented by $\int_0^a f(x)dx$, $\int_a^b f(x)dx$, and $\int_0^b f(x)dx$. Write an equation relating those 3 integrals [two of them add up to the third]. Solve the equation for $\int_a^b f(x)dx$ and use your general formula from Exercise 7 twice to guess the **most general formula**. $\int_a^b f(x)dx =$_____ *(Exercise 9 may help.)*

Exercise 9: There is some chance that Exercises 7 and 8 have left you puzzled. It’s a **big** step to make this generalization! In any event, go to the Data/Matrix Editor again, put the same 4 functions in column **c1**, and change the formula stored in **c2**. Instead of having **c2** compute $\int_0^b c1(x)dx$, make it compute $\int_a^b c1(x)dx$. [To do so, highlight **c2** as shown in figure 1, press [ENTER], and type· **(c1,x,a,b)**[ENTER].] Maybe filling in the data grid in figure 5 will help to correctly answer the questions in Exercises 7 and 8 or to <u>confirm</u> those answers.

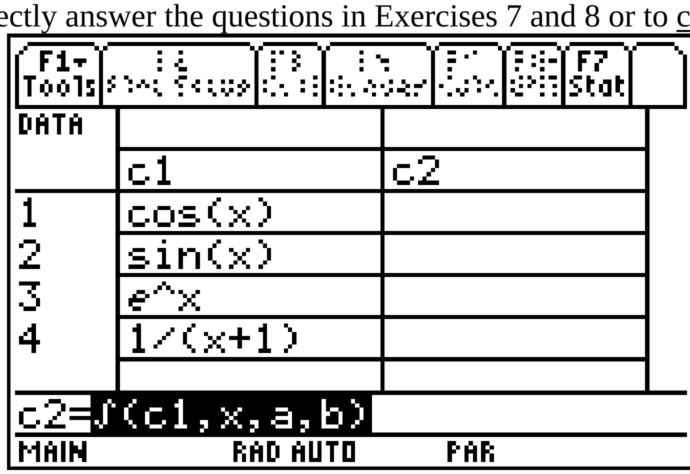


Fig.5

Exercise 10: If you still don't feel confident about your general formulas in Exercises 7 through 9, here's the difficult-to-predict fact that you've been trying to see:

<u>FACT</u>

In order to find the <u>exact</u> value of $\int_a^b f(x)dx$,

you must first find a function G whose derivative is f

and then compute $G(b) - G(a)$.

(By the way, f must be *continuous* on $[a,b]$, a very important hypothesis.)

A function such as G in the box above is called an ***anti**derivative* of f. That is, if the derivative of G is f, then G is called **an** antiderivative of f. (In symbols, if $G'(x) = f(x)$ then G is called an antiderivative of f.)

For the data grid in figure 6, which is actually the filled-in data grid for Exercise 9, confirm that the column **c2** results are what the "FACT" above says should be there. That is, find the antiderivative of each function in column **c1**, plug in b and a, and subtract. [Confirm this by taking the derivative of the antiderivative. You should get the function in the integrand (the function in column **c1**.)] (Note that the '89's screen is too narrow to see the entire contents of any cell in column **c2** except for row 3, but you can highlight each of the other rows, as shown, and read the entire contents in the command line at the bottom of the screen.)

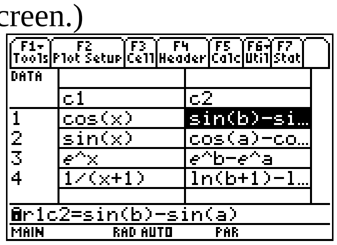


Fig. 6

The "FACT" above is often called ***The First (or Second) Fundamental Theorem of Calculus***. What it is called varies a lot from book to book, from class to class. Needless to say, by any name, it is <u>very important</u>, as it encapsulates a link between derivatives and integrals!