The Painted Cube

Math Objectives

- Students will explore constant, linear, quadratic, and cubic functions. The functions will be modeled from numerical data that they generate by thinking of an $n \times n \times n$ cube being dipped in paint, and how many of the cubes have paint on zero, one, two, or three faces.
- Model with mathematics. (CCSS Mathematical Practice)
- Construct viable arguments and critique the reasoning of others. (CCSS Mathematical Practice)

Vocabulary

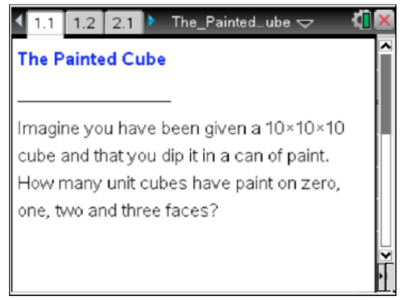
- regression
- constant, linear, quadratic, cubic
- scatter plot
- factored form of a polynomial equation

About the Lesson

- This lesson involves having the students hypothesize about the different relationships that exist between the size of the cube and the number of cubes that have paint on one, two, three, and zero faces. In order to help students visualize the problem, interlocking cubes could be made available.
- As a result, students will model the relationships between the cube size and the number of painted faces by looking at the graphical representations and then creating algebraic models.

TI-Nspire™ Navigator™ System

- Use **Screen Capture** to monitor the models that students are using for each of the different problems.
- Use **Live Presenter** as needed to model the manipulation of the movable line.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire™ document
- Open a document
- Move between pages
- Add and manipulate a moveable line

Tech Tips:

- Make sure the font size on your TI-Nspire™ handhelds is set to Medium.

Lesson Materials:

Student Activity

- The_Painted_Cube_Student.pdf
- The_Painted_Cube_Student.doc

TI-Nspire™ document

- The_Painted_Cube.tns

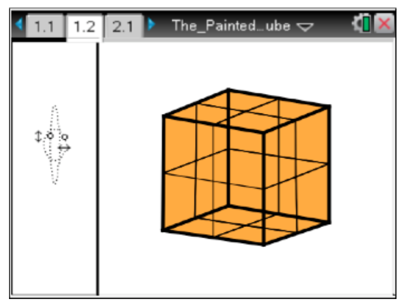
Lesson Procedures, Discussion Points, and Possible Answers

Tech Tip: If students experience difficulty dragging a point, check to make sure that they have moved the arrow until it becomes a hand ( ) getting ready to grab the point. Press ctrl to grab the point and close the hand ( ).

Part 1—Introducing the Problem

Press ctrl ▶ and ctrl ◀ to navigate through the lesson.

Move to page 1.2.



One strategy for solving a problem is to solve a simpler related problem.

1. Consider a 2 × 2 × 2 cube.

   a. How many unit cubes does it take to build a 2 × 2 × 2 cube?

   Answer: 8 unit cubes

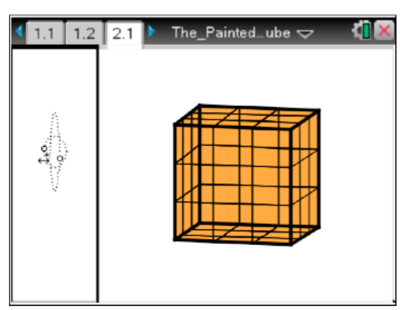
   b. Rotate the model on page 1.2 by dragging the open points on the left side of the screen. If needed, build your own model using cubes. If this cube were dipped in paint, what is the greatest number of faces of a single unit cube that could be painted?

   Answer: The greatest number of faces that could be painted is three.

   c. How many faces of *each* of the unit cubes are painted on the 2 × 2 × 2 cube?

   Answer: All cubes have three faces painted.

Move to page 2.1.



2. Now consider a 3 × 3 × 3 cube.

   a. How many unit cubes does it take to build a 3 × 3 × 3 cube?

   Answer: 27 cubes

b. Rotate the model on page 2.1 by dragging the open points on the left side of the screen. If needed, build your own model using cubes. If the 3 × 3 × 3 cube were dipped in a can of paint, how many faces of *each* of the unit cubes would be painted?

Answer: 8 cubes would have three faces painted; 12 cubes would have two faces painted; 6 cubes would have one face painted; 1 cube would have zero faces painted.

3. a. Record your findings for the 2 × 2 × 2 and 3 × 3 × 3 cubes in the table below. Then determine how many faces of each of the unit cubes would be painted for the 4 × 4 × 4 and 5 × 5 × 5 cubes if the large cubes were dipped in paint.

n (side length of cube)	**Number of unit cubes with paint on zero faces**	**Number of unit cubes with paint on one face**	**Number of unit cubes with paint on two faces**	**Number of unit cubes with paint on three faces**
2	0	0	0	8
3	1	6	12	8
4	8	24	24	8
5	27	54	36	8

b. What patterns do you notice in the table?

Sample Answer: For any value of n where $n \geq 2$, there will always be 8 cubes that have three faces painted.

The number of cubes with two faces painted is always a multiple of 12. The number of unit cubes with paint on two faces increases by 12 each time.

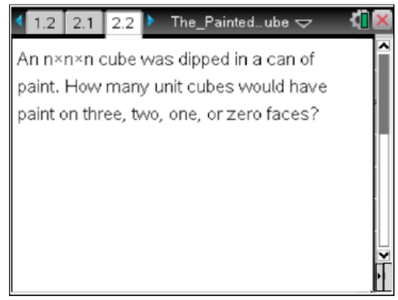
For the number of cubes with one face painted, the number increases by 6, then 18, then 30. There is a difference of 12 between each of those numbers.

The number of unit cubes with paint on zero faces is a perfect cube. It is the cube of the number that is two less than the side length of the cube.

Teacher Tip: Some students might already see functional relationships, but the expectation at this point is that they at least see some kind of pattern in the numbers.

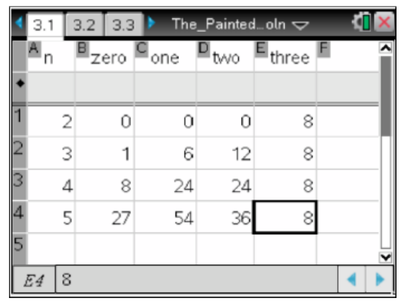
Part 2—Investigating Paint on Three Faces

Move to page 2.2.



You will now analyze the data you collected and explore the relationships graphically for a cube with any side length *n*. You will then use the graph to make predictions for the case where *n* = 10.

Move to page 3.1.



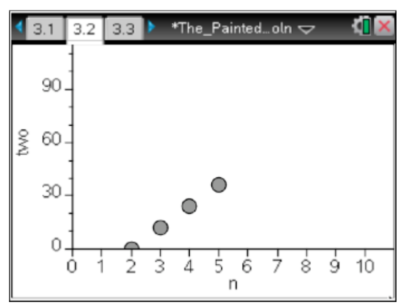
4. Enter the values from your table above into the spreadsheet on page 3.1.

5. From the information in the table, how many unit cubes would have paint on <u>three</u> faces in a 10 × 10 × 10 cube? Explain your reasoning.

   <u>Answer:</u> There will be 8 cubes with paint on three faces. The only cubes that can have paint on three faces are the corner cubes, and there will always be 8 of them.

Part 3—Investigating Paint on Two Faces

Move to page 3.2.



This page uses the data that you entered on page 3.1 to make a scatter plot of the number of unit cubes with paint on two faces versus the side length of the cube, *n*.

6. Describe the relationship between the two variables.

   <u>Answer:</u> The data looks linear.

7. Add a movable line by selecting **Menu > Analyze > Add Movable Line**.

   a. Grab the line and transform it to get a line of best fit. What is the equation of your line of best fit?

      <u>Answer:</u> **m1**(*x*) = 12*x* – 24 or **f**(*n*) = 12*n* – 24

> **Tech Tip:** Moving the cursor near the middle of the movable line will display the translate tool. Moving the cursor on either end of the movable line will display the rotation tool. Press ctrl to grab the line and translate or rotate it.

b. Test your equation with known values from your table and adjust your movable line as necessary. Once your equation matches the known values, what is the equation of your line?

Answer: **f**(n) = 12n – 24

> **Tech Tip:** Students can use the **Scratchpad** as needed for calculations.

c. Write your equation in factored form. What is the meaning of this form of the equation in the context of the painted cube problem?

Answer: **f**(n) = 12n – 24 = 12(n – 2)

The cubes that have paint on two faces are the cubes on the edges of the large cube, excluding the corners. There are 12 edges on a cube. Subtracting 2 removes the corner cubes.

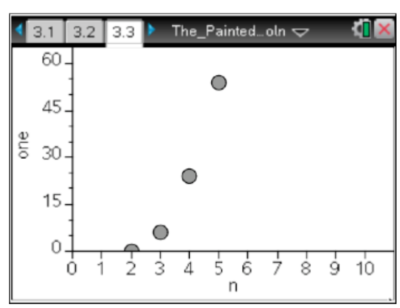
8. Use your equation to determine the number of unit cubes that would have paint on two faces in a 10 × 10 × 10 cube.

   Answer: 96 unit cubes

9. Explain how your answer makes sense in terms of the graph on page 3.2.

   Answer: The point (10, 96) would be on the line.

Part 4—Investigating Paint on One Face



Move to page 3.3.

This page uses the data that you entered on page 3.1 to make a scatter plot of the number of unit cubes with paint on one face versus the side length of the cube, n.

10. Describe the relationship between the two variables.

    Answer: The data looks quadratic.

11. Determine the equation of the curve of best fit. Press **Menu > Analyze > Regression** and select the type of function that you think will best fit the data.

a. What is the regression equation?

Answer: $y = 6x^2 – 24x + 24$

b. Test your equation with known values from your table. If needed, choose a different type of regression equation. Once the equation matches the known values, what is the equation?

Answer: $y = 6x^2 – 24x + 24$ or $\mathbf{f}(n) = 6n^2 – 24n + 24$

c. Write your equation in factored form. What is the meaning of this form of the equation in the context of the painted cube problem?

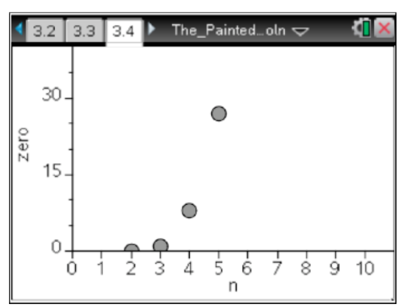
Answer: $\mathbf{f}(n) = 6n^2 – 24n + 24 = 6(n^2 – 4n + 4) = 6(n – 2)(n – 2) = 6(n – 2)^2$

The cubes that have paint on one face are the cubes on the faces of the large cube, excluding the edges. There are six faces on a cube. On the face of an $n \times n \times n$ cube, the number of squares in the center (excluding the edges) is $(n – 2)^2$.

12. Use your equation to determine the number of unit cubes that would have paint on one face in a $10 \times 10 \times 10$ cube.

Answer: 384 unit cubes

Part 5—Investigating Paint on Zero Faces



Move to page 3.4.

This page uses the data that you entered on page 3.1 to make a scatter plot of the number of unit cubes with paint on zero faces versus the side length of the cube, n.

13. Describe the relationship between the two variables.

Answer: The data looks cubic.

14. Determine the equation of the curve of best fit. Press **Menu > Analyze > Regression** and select the type of function that you think will best fit the data.

    a. What is the regression equation?

    Answer: $y = x^3 - 6x^2 + 12x - 8$ or $\mathbf{f}(n) = n^3 - 6n^2 + 12n - 8$

    b. Test your equation with known values from your table. If needed, choose a different type of regression equation. Once the equation matches the known values, what is the equation?

    Answer: $\mathbf{f}(n) = n^3 - 6n^2 + 12n - 8$

15. Use your equation to determine the number of unit cubes that would have paint on zero faces in a $10 \times 10 \times 10$ cube?

    Answer: 512 unit cubes

Part 6—Reflecting on the Problem

16. a. Record the type of relationship (e.g., linear, quadratic) for each of the numbers of painted faces you investigated.

Painted Faces	Type of Relationship
3	Constant
2	Linear
1	Quadratic
0	Cubic

    b. Think about the painted cubes and how the numbers of painted faces change as the side length of the cube, *n*, increases. Justify why each type of relationship makes sense in the context of the problem.

    Sample Answer: As *n* increases:

- The number of corner cubes stays constant: 3 faces painted
- The edges of the cube grow constantly (linear relationship): 2 faces painted
- The faces of the cube grow in area (quadratic relationship): 1 face painted
- The middle of the cube (excluding the faces) grows in volume (cubic relationship): 0 faces painted.

Extension—Making Sense of the Cubic Relationship

The regression formula for the cubic relationship, **f**(n) = $n^3 - 6n^2 + 12n - 8$, is not as easy to understand in the context of the problem. Ask students to look back at their table on the Student Worksheet, specifically at the column for zero faces painted. Lead class discussion with questions such as these:

1. Looking at the table on page 1, what kind of numbers are in the zero column?

   Answer: They are perfect cubes.

2. Write these numbers as powers of three.

   $0 = 0^3$, $1 = 1^3$, $8 = 2^3$, $27 = 3^3$,

3. How do the bases of these numbers relate to n?

   Answer: The bases are two less than n.

4. How else could we express the number of cubes with zero faces painted as a function of n?

   Answer: f(n) = $(n - 2)^3$

5. How does this formula compare to the one found in question 14b?

   Answer: $(n - 2)^3 = (n - 2)(n - 2)^2 = (n - 2)(n^2 - 4n + 4) = n^3 - 6n^2 + 12n - 8$ ✓