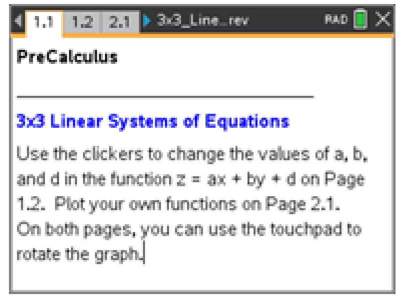
3x3 Linear Systems of Equations



Math Objectives

- Students will be able to describe the effects of the coefficients of a linear function in three variables on the graph of the function.
- Students will be able to identify the number of solutions to a system of linear equations in three variables by analyzing the graphs of the equations.
- Students will be able to describe the conditions under which a system of linear equations in three variables will have 0, 1, or infinitely many solutions.
- Students will construct viable arguments and critique the reasoning of others (CCSS Mathematical Practice).
- Students will look for and make use of structure (CCSS Mathematical Practice).

Vocabulary

- system of equations
- plane
- solution to a system of equations

About the Lesson

- This lesson involves connecting graphical representations of systems of linear equations in three variables to the number of solutions of those systems.
- As a result, students will:
  - Manipulate a plane to observe the effect of the coefficients on the graph of the plane.
  - Plot systems of two linear functions in three variables to investigate the graphical representations of solutions of a system.
  - Plot systems of three linear functions in three variables to investigate the graphical representations of solutions of a system.
  - Determine the number of solutions a system of linear equations in three variables can have.

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to examine patterns that emerge.
- Use Quick Poll to assess students' understanding.

TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Files:

Student Activity

3x3_Linear_Systems_of_Equations_Student.pdf

3x3_Linear_Systems_of_Equations_Student.doc

TI-Nspire document

3x3_Linear_Systems_of_Equations.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

1. When solving a system of two linear equations with two variables, what do you look for graphically to indicate the solution? Explain.

   Sample Answers: One looks for the intersection point of the lines to indicate the solution of the system. The intersection indicates a point that satisfies both equations simultaneously. The intersection point lies on the graph of both functions, and therefore the coordinates of the point satisfy the system. Moreover, if two lines do not intersect, that indicates that the system has no solutions.

2. A linear equation in three variables is an equation of the standard form $ax + by + cz = d$. What do you think the graph of such a function might look like? Why?

   Sample Answers: Student predictions will vary. Some students might suggest that it is still a line, others might be familiar with planes, and others might predict that it is a more complicated function in 2-space because of the added variable.

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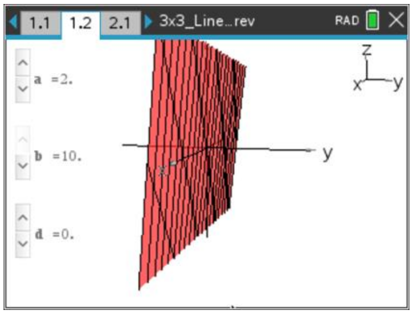

3. This page shows a graph of a linear function, $z = ax + by + d$. The graph is called a plane.
   a. What is the equation in standard form of the plane shown on the page?

   Sample Answers: $2x + 10y - z = 0$

   b. Use the clicker to change the value of *a*. What happens to the plane as *a* changes? What attribute of the plane does *a* affect?

   Sample Answers: As *a* changes, the angle that the plane makes with the horizontal plane changes. The value of *a* is the slope of the plane in the x-direction. That is, if the other coefficients are held constant, *a* gives the rate of change of *z* with respect to *x*.

c. Use the clicker to change the value of *b*. What happens to the plane as *b* changes? What attribute of the plane does *b* affect?

Sample Answers: The value of *b* is the slope of the function in the *y* direction. That is, if the other coefficients are held constant, *b* is the rate of change of *z* with respect to *y*.

> **Teacher Tip:** Note that the directional rate of change interpretations for *a* and *b* might be subtle for students to observe. Teachers can help expose this idea by leading students through fixing one independent variable and observing the resulting change in the dependent variable given change in the other independent variable.

d. Use the clicker to change the value of *d*. What happens to the plane as *d* changes? What attribute of the plane does *d* affect?

Sample Answers: The value of *d* changes the vertical intercept of the plane. This also changes the two horizontal intercepts of the plane.

> **Teacher Tip:** Students might notice that the plane appears to rotate about the vertical axis when *a* and *b* are changed. In order to help students make the connection between the coefficient and directional slope, the teacher can suggest that students set the other coefficients to 0, and/or experiment with different viewing perspectives for the graph. Students might also notice that changing *c* appears to shift the graph horizontally. Teachers might want to remind students of a similar phenomenon with linear functions in two variables, and that a horizontal shift and vertical shift are analogous due to the fact that lines and planes are unbounded.

> **Tech Tip:** Use the up, down, left, and right arrows in the software or on the handheld's touchpad to rotate the graph for a different perspective.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 1 at the end of this lesson.

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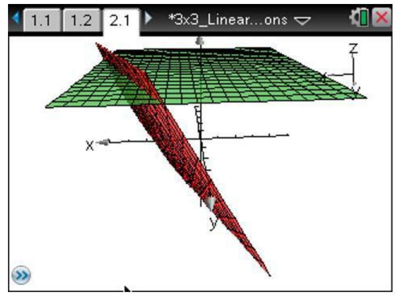


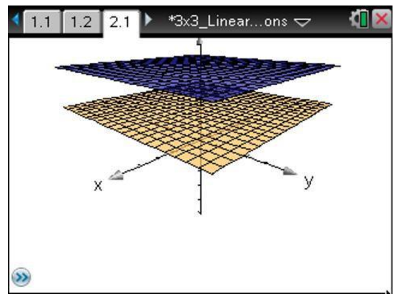
4. Without graphing, how many solutions do you think the following system will have?

   $z = 2$

   $z = 4$

   Explain your reasoning.

   Sample Answers: The system will have 0 solutions. Both are equations of horizontal planes, and so they will not intersect.



5. If any functions are plotted, clear them, and then plot the two functions from Question 4. How many solutions does the system have? How do the graphs help you see this?

   Sample Answers: The system does not have any solutions. The graphs show that the planes do not intersect, and therefore share no points in common. This indicates that no points will satisfy both equations, and so the system has no solutions.

> **Tech Tip:** Students who are unfamiliar with the 3-D graphing feature might need some help plotting functions. Teachers might want to explain to students that they should click on the chevron to expand the function entry line and then enter the functions as z1 and z2, in the same way that functions are plotted in traditional graphing.

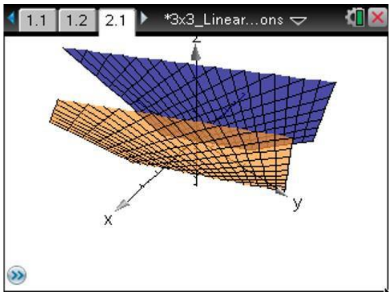
6. Without graphing, how many solutions do you think the following system will have?

   $z = 2x + y - 3$

   $z = 2x + y + 4$

   Can you use what you know about systems of linear equations in two variables to help you?

   Sample Answers: The system will have 0 solutions. The planes are parallel, having the same rate of change in the *x*-direction and the same rate of change in the *y*-direction. A system of linear equations in two variables with no solutions has graphs which are non-intersecting (parallel) lines. These functions have the same slopes, so one could predict that linear functions in three variables having equal rates of change with respect to each of the independent variables would also be non-intersecting.



7. Clear the plotted functions, and plot the two functions from Question 6. How many solutions does the system have? How do the graphs help you see this?

Sample Answers: The system has no solutions. The graphs are parallel planes, indicating that no point will satisfy both equations simultaneously.

8. Without graphing, how many solutions do you think the following system will have?

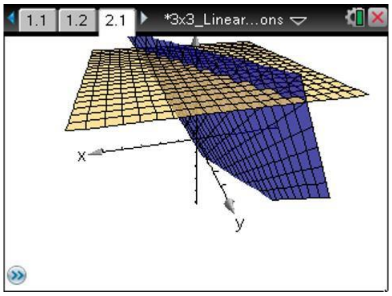
$$z = 4$$
$$z = 2x + y + 4$$

Explain your reasoning.

Sample Answers: The system will have infinitely many solutions. The first equation is satisfied by any point (x, y, 4). In order for such point to satisfy the second equation, it must be true that $4 = 2x + y + 4$, so $2x + y = 0$. Therefore, any point on the line with coordinates (x, $2x$, 4) satisfies both equations. This means that the system has infinitely many solutions.

9. Clear the plotted functions, and plot the two functions from Question 8. How many solutions does the system have? How do the graphs help you see this?

Sample Answers: The system has infinitely many solutions. The planes intersect in a line, so every point on that line will satisfy both equations.



10. Clear the plotted functions, and plot several other pairs of functions to investigate. Based on your investigations, how many solutions do you think a system of two linear equations in three variables can have? Explain your reasoning.

Sample Answers: A system of two linear equations in three variables can either have no solutions or infinitely many solutions. Two planes might not intersect, might intersect in a line, or might be the same planes. If two planes do not intersect, there will be no solutions to the system. If they intersect in a line, or if they are the same planes, there are infinitely many solutions.

11. Now suppose you have three equations in three variables. Based on your work with two functions, predict how many possible solutions a system of three linear equations can have. Justify your prediction geometrically.

    Sample Answers: A system of three linear equations in three variables can either have no solution, one solution, or infinitely many solutions. If the three planes do not intersect in a common point or common line, the system does not have a solution. If the three planes intersect in a common point, the system has one unique solution. If the three planes intersect in a line, or if all three are the same plane, the system has infinitely many solutions.

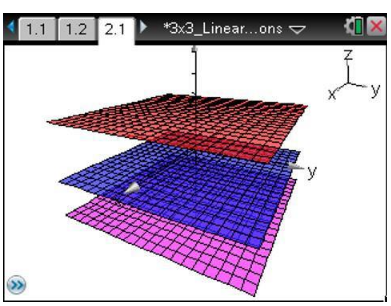
12. Clear the plotted functions, and plot several groups of three functions to investigate. How many solutions can a system of three linear equations in three unknowns have? How can you see these in the graphs?

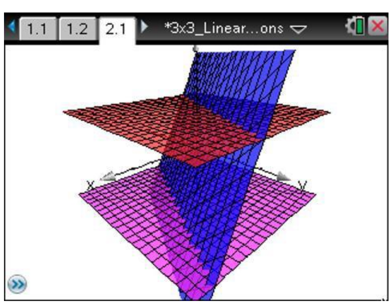
    Sample Answers: See the answer to Question 11.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 2 at the end of this lesson.

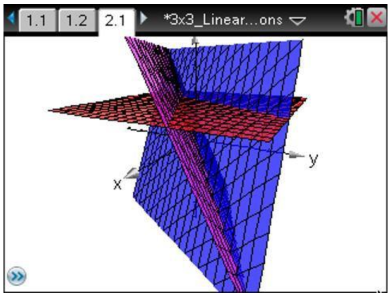
Teacher Tip: Examples of the cases are pictured below for teacher reference.

No Solutions:

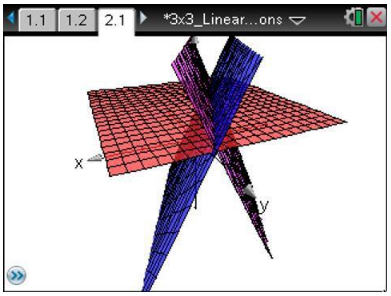




One Solution:



Infinitely Many Solutions:



Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- The connection between the graph of a linear function in three variables and its parameters.
- That a system of two linear equations in three variables will not have a unique solution.
- That a system of three linear equations in three variables might have 0, 1, or infinitely many solutions.
- The connections between the graphs of three linear functions in three variables and the number of solutions of the system.

Assessment

Teachers can give students systems to solve, with particular focus on predicting and testing predictions about the number of solutions based on the graphs of the functions.

TI-Nspire Navigator

Note 1

Question 3d, Name of Feature: Quick Poll

A Quick Poll can be used to dynamically share students' observations regarding the impact of parameters on the graph of a plane.

Note 2

Question 12, Name of Feature: Screen Capture

A Screen Capture can be used to show all the different possible arrangements of three planes and launch discussions about what these arrangements reveal about the number of solutions of a system.