



THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

TI-89 Workshop

Algebra and Calculus



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1. Saying 'Hello' to your CAS calculator

You will use the following keys.

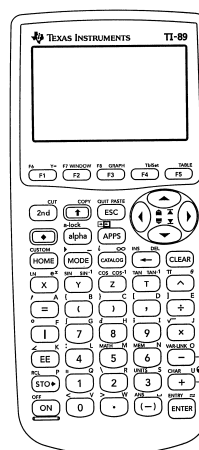
- Press **[ON]**

The calculator cursor should be in the Home Screen (see the black cursor flashing in the bottom left hand corner).

- Press **[2nd]** **[ON]**

The calculator should turn off.

- If you can't see the screen use **[◀]** **[▶]** (lighter) or **[◀]** **[▶]** (darker) to change screen contrast.
- [HOME]** displays the Home Screen, where you perform most calculations.

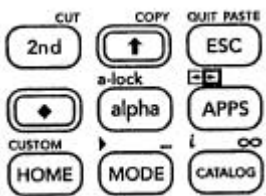


Basic Facilities of the TI-89

Function Keys	Cursor Pad
<p>[F1] through [F8] function keys let you select toolbar menus.</p>	<p>The cursor is controlled by the large blue circle on the top right hand side of the calculator. This allows access to any part of an expression.</p>
Application Short Keys	Calculator Keypad
<p>Used with the [◀] key to let you select commonly used applications: [Y=] [WINDOW] [GRAPH] [TblSet] [TABLE]</p>	<p>Performs a variety of mathematical and scientific operations</p>

[2nd] **[◀]** **[▶]** and **[ALPHA]** modify the action of other keys:

Modifier	Description
[2nd] (Second)	Accesses the second function of the next key you press
[◀] (Diamond)	Activates “shortcut” keys that select applications and certain menu items directly from the keyboard.
[▶] (Shift)	Types an uppercase character for the next letter key you press.
[ALPHA]	Used to type alphabetic letters, including a space character. On the keyboard, these are printed in the same colour as the [ALPHA] key.

	Key	Description
	APPS	Displays a menu that lists all the applications available on the TI-89.
	ESC	Cancels any menu or dialogue box.
	ENTER	Evaluates an expression, executes an instruction, selects a menu item etc...
	MODE	Displays a list of the TI-89's current mode settings, which determine how numbers and graphs are interpreted, calculated, and displayed.
	CLEAR	Clears (erases) the entry line.
	CATALOG	Press \odot or \ominus to move the cursor to the function or instruction. (You can move quickly down the list by typing the first letter of the item you need.) Press ENTER Your selection is pasted on the home screen.

Application	Lets you:
[Home]	Enter expressions and instructions, and performs calculations
[Y=]	Define, edit, and select functions or equations for graphing
[Window]	Set window dimensions for viewing a graph
[Graph]	Display graph
[Table]	Display a table of variable values that correspond to an entered function

Press:	To display
F1 F2 ... etc.	A toolbar menu– Drops down from the toolbar at the top of most application screens. Lets you select operations useful for that application
2nd [CHAR]	CHAR menu– Lets you select from categories of special characters (Greek, math, etc.)
2nd [MATH]	MATH menu– Lets you select from categories of mathematical operations

- **2nd** [F6] Clean Up to start a new problem:

Clear a–z Clears (deletes) all single-character variable names in the current folder.
 If any of the variables have already been assigned a value, your calculation may produce misleading results.

Problem?	Try this!
If you make a typing error	If you make a typing error use \leftarrow to undo one character at a time If necessary, press CLEAR to delete the complete line.
If you want to clear the home screen completely	Press F1 [8]

Mode Settings

Press **[MODE]**, this shows the modes and their current settings



If you press **F2** then 'Split Screen' specifies how the parts are arranged: FULL (no split screen), TOP-BOTTOM, or LEFT-RIGHT



(a) Entering a Negative Number

Use **[−]** for subtraction and use **[(-)]** for negation.

To enter a negative number, press **[(-)]** followed by the number.

To enter the number -7 , press **[(-)] 7**.

$$9 \times (-) 7 = -63,$$

$9 \times [-] 7$ = displays an error message

To calculate $-3 - 4$, press **[(-)] 3 [-] 4 [ENTER]**

(b) Implied Multiplication

If you enter: The TI-89 interprets it as:

$2a$ $2*a$

xy Single variable named xy ; CAS does not read as $x \times y$

(c) Substitution

Using **[|]** key to find the value of a function or expression given particular values of a variable

eg) $x^2 + 2 [|] x = 3$

This calculates the value of $x^2 + 2$ given $x = 3$

Using 'STORE' key: **[STO▶]**

eg) Find $f(2)$ if $f(x) = -x^3 + 2$

$$-x^3 + 2 \text{ [STO▶] } f(x)$$

$$f(2)$$

$$-x^3 + 2 \rightarrow f(x)$$

$$-6$$

(d) Rational Function Entry

$$\frac{f(x)}{g(x)} = \frac{(f(x))}{(g(x))} = (\text{numerator}) \div (\text{denominator})$$

For example, $\frac{x+1}{2x-1} \rightarrow (x+1) \div (2x-1)$

(e) Operators

addition: + subtraction: − multiplication: × division: ÷ Exponent: ^

(f) Elementary Functions

exponential: e^x natural logarithm: $\ln(x)$ square root: $\sqrt{}$ absolute value: $\text{abs}(x)$

trigonometric: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$

If you want $\sec(x)$ then put $1/\cos(x)$ or use the catalogue: **[CATALOG] [3] [ENTER]**, $\text{cosec}(x)$ is $1/\sin(x)$.

Note: The trigonometric functions in TI-89 angles are available in both degrees and radians. If you want degrees (180°) or radians (π) change using the 3 key previously discussed.

(g) Constants

i : imaginary number

π : Pi

∞ : infinity

with **[2nd] [CATALOG]** key

with **[2nd] [^]** key

with **[♦] [CATALOG]** key

(h) Recalling the last answer

$\boxed{2\text{nd}} \boxed{[\text{ANS}]}$

ex) ans(1) Contains the last answer

ans(2) Contains the next-to-last answer

(i) Cutting, Copying and Pasting

Use $\odot \odot$ or $\odot \odot$ to highlight an expression.

Press $\boxed{[F1] 5}$, to copy and $\boxed{[F1] 6}$ to paste.

Press $\boxed{[ENTER]}$ to replace the contents of the entry line with any previous entry.

(j) When differentiating with respect to x

Limit $\lim_{x \rightarrow a} f(x)$: $\text{lim}(f(x), x, a)$

Differentiation $\frac{d}{dx} f(x)$: $d(f(x), x)$

Indefinite Integral $\int f(x) dx$: $\int (f(x), x, c)$

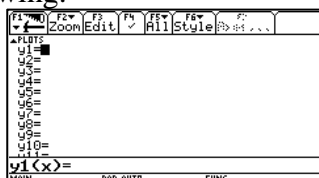
Definite integral $\int_a^b f(x) dx$: $\int (f(x), x, a, b)$

Area between $f(x)$ and $g(x)$ on the interval $[a, b]$: $\int_a^b |f(x) - g(x)| dx$

2. $\boxed{[Y=]}$ and $\boxed{[Table]}$

(a) The $\boxed{[Y=]}$ menu

Press $\boxed{\blacklozenge} \boxed{[Y=]}$ to see the following:



If there are any functions to the right of any of these eight equal signs, place the cursor on them (using the arrow keys) and press $\boxed{[CLEAR]}$

Place the cursor just to the right of $y1=$ and follow the sequence below.

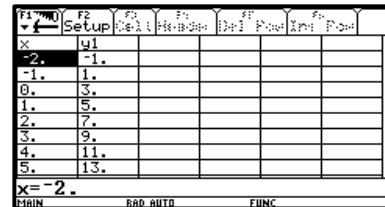
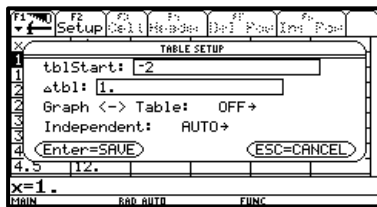
Press	See	Explanation
$2x + 3$	$y1(x) = 2x + 3$	You have entered $y1 = 2x + 3$
$\boxed{[HOME]}$		This returns you to a blank Home Screen.
$y1(x) \boxed{[ENTER]}$	$y1(x)$ $2x + 3$	This pastes $y1$ on the Home Screen.
$y1(4) \boxed{[ENTER]}$	$y1(4)$ 11	This finds the value of $y1$ when $x = 4$.

(b) Table

Press $\boxed{\blacklozenge} \boxed{[TABLE]}$ to see the table of values for $2x + 3$, as shown below:

Press $\bullet \boxed{[TblSet]}$, change the settings and see the effect in $\boxed{[TABLE]}$.

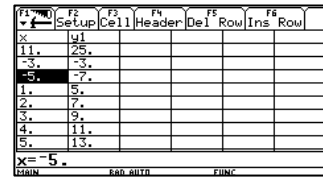
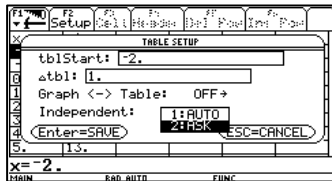




By changing [TblSet] from [1. AUTO] to [2. ASK], complete the table below:

x	y_1
11	?
-3	?
-5	?

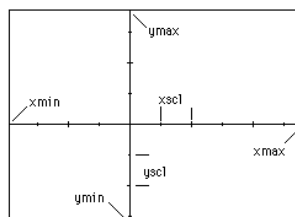
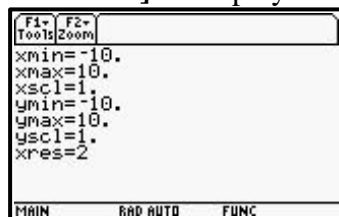
Remember: y_1 is still set to $2x + 3$



3. Graphing

(a) Displaying Window Variable in the Window Editor

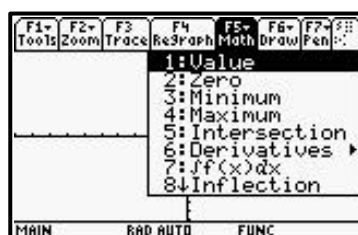
Press **◆** **[WINDOW]** to display the Window Editor.



Variables	Description
xmin, xmax, ymin, ymax	Boundaries of the viewing window.
xscl, ysc1	Distance between tick marks on the x and y axes.
Xres	Sets pixel resolution (1 through 10) for function graphs. The default is 2.

(b) Overview of the Math Menu

Press **F5** from the Graph screen

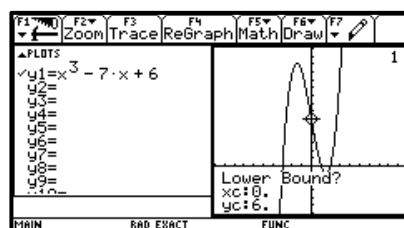
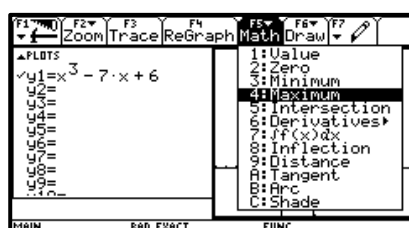
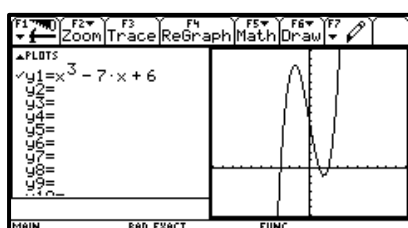


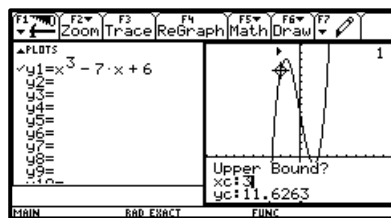
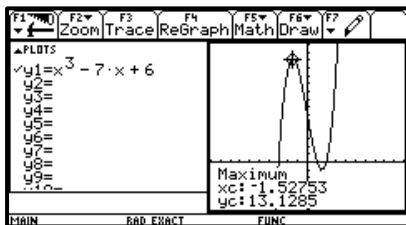
Math Tool	Description
Value	Evaluates a selected $y(x)$ function at a specified x value
Zero, Minimum, Maximum	Finds a zero (x -intercept), minimum, or maximum point within an interval.
Intersection	Finds the intersection of two functions.
Derivatives	Finds the derivative (slope) at a point.
$\int f(x)dx$	Finds the approximate numerical integral over an interval.
A:Tangent	Draws a tangent line at a point and displays its equation

(c) Finding the Maximum & Minimum Values of a Function from its Graph

1. Display the **Y=Editor**.
2. Enter the function
3. Open the Math Menu **F5**, and select **4: Maximum**.
4. Set the lower bound.
5. Set the upper bound.
6. Find the maximum point on the graph between the lower and upper bounds.
7. Transfer the result to the Home screen, and then display the Home screen.

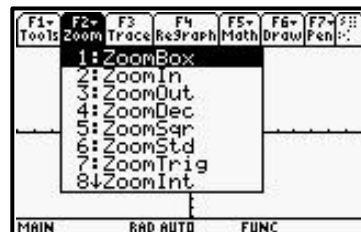
[HOME]





(d) Overview of the Zoom Menu

Press **F2** from **y=Editor**, window Editor, or Graph screen



Zoom tool	Description
1:ZoomBox	Lets you draw a box and zoom in on that box.
2:ZoomIn 3:ZoomOut	Lets you select a point and zoom in or out by an amount defined by SetFactors .
4:ZoomDec	Sets Δx and Δy to 0.1, and centres the origin.
6:ZoomStd	Sets Window variables to their default values. $x_{\min} = -10, x_{\max} = 10, x_{\text{scl}} = 1, y_{\min} = -10, y_{\max} = 10, y_{\text{scl}} = 1, x_{\text{res}} = 2$

Notes:

To get out of the graphing mode use **[HOME]**.

This will not work while the **BUSY** icon is flashing in the bottom right hand corner.

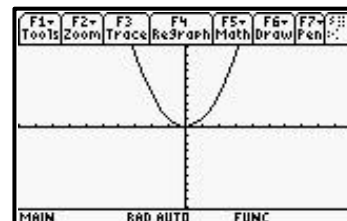
Adjust your graph by selecting **F2** and choosing **2:ZoomIn**, **3:ZoomOut**, or **A:ZoomFit**

eg) Graph $y = x^2$ by following these instructions.

◆ **[y=]** x^2 **[ENTER]**



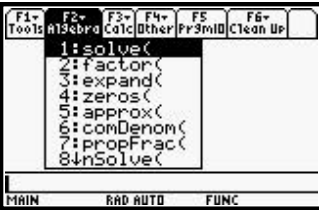
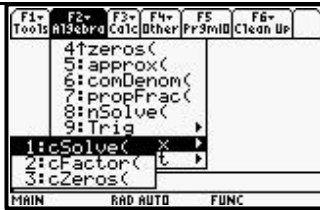
◆ **[F3]**



To draw a new graph go to **[Y=]** and change the formula in the **y1** position using the cursor to move up to it to delete it. This effectively clears the previous graph as well. Alternatively, using **y2** will add the new graph to $y = x^2$.

[HOME] returns you to the Home screen.

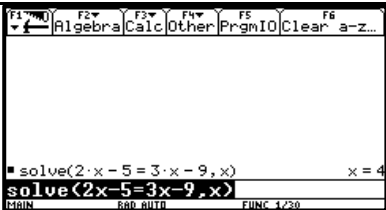
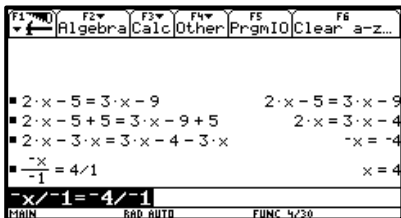
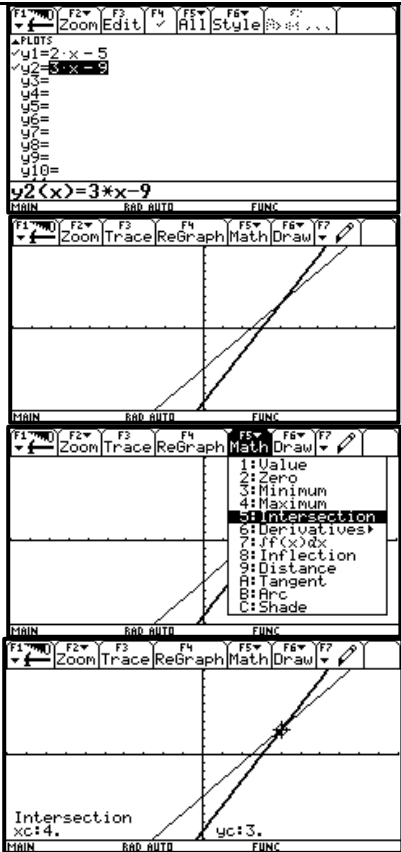
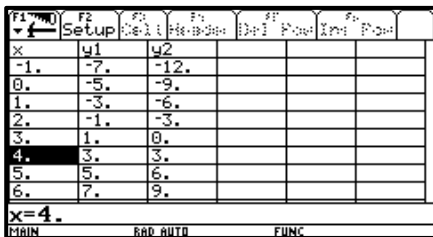
4. The Algebra Menu

Menu Item	Description F2 MENU	 
1: solve	Solves an expression for a specified variable. This returns solutions only, regardless of the Complex Format mode setting (For complex solutions, select A:Complex from the algebra menu).	
2: factor	Factorises an expression with respect to all its variables or with respect to only a specified variable.	
3: expand	Expands an expression with respect to all its variables or with respect to only a specified variable.	
4: zeros	Determines the values of a specified variable that make an expression equal to zero.	
5: approx	Evaluates an expression using floating-point arithmetic, where possible.	
6: comDenom	Calculates a common denominator for all terms in an expression and transforms the expression into a reduced ratio of a numerator and denominator.	
7: propFrac	Returns an expression as a proper fraction expression.	

Solving Linear Equations

Example. Solve $2x - 5 = 3x - 9$.

We can solve this in three different ways: algebraically, graphically, and numerically.

Press	See	Explanation
Method 1 a) [HOME] F2 1 $2x - 5 = 3x - 9, x)$ [ENTER]		$2x - 5 = 3x - 9$ is solved by an algebraic method. The , x tells the calculator to solve with respect to x. $x = 4$ is the value which makes both sides equal in value.
Method 1 b) $2x - 5 = 3x - 9$ [ENTER] $2x - 5 + 5 = 3x - 9 + 5$ [ENTER] $2x - 3x = 3x - 3x$ [ENTER] $-x / -1 = -4 / -1$ [ENTER]		To find the value of x, we need to simplify the given expression step by step: If we add 5 to both sides, the expression is simplified to $2x = 3x - 4$. If we subtract $3x$, the expression is simplified to $-x = -4$. If we divide by -1 , finally we get $x = 4$
Method 2. ♦ [F1] $2x - 5$ [ENTER] $3x - 9$ [ENTER] ♦ [F3] [F5] 5 1 st curve? [ENTER] 2 nd curve? [ENTER] Lower bound? 0 [ENTER] Upper bound? 6 [ENTER]		Here each side of the equation is defined as a function, using $y_1(x)$ and $y_2(x)$: $y_1(x) = 2x - 5$ $y_2(x) = 3x - 9$ Looking at the two graphs, we can see that they intersect at one point. To find the intersection point we need to use the function key [F5]. 1 st curve means $y_1(x)$, 2 nd curve means $y_2(x)$. The lower and upper bound means the interval in which the intersection point is found. So the two graphs intersect at the point (4, 3). i.e. $x = 4$
Method 3. ♦ [F4] tblStart: -1 Δ tbl: 1 ♦ [TABLE]		The point of intersection can be found using a table. Enter y_1 and y_2 as in method 2. When we look at the point $x=4$, we can see the values of the two functions are the same, and equal to 3.

Looking at the three methods we see that the value of x is the same in each case.

Exercise

Solve the following equations. Make sure you use each of the three methods above at least once.

1. $|3x - 2| = 5$

2. $x^2 - 2x + 7 = 22$

3. $\sqrt{2-x} = x$

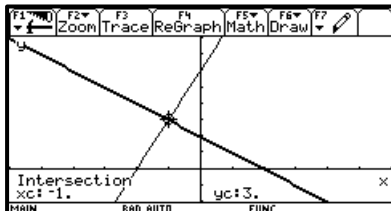
4. $\ln\left(\frac{x+1}{2}\right) - \ln\left(\frac{x}{2}\right) = 3$

5. $e^{4x} = 4^{3-2x}$
(give the exact solution)

Screen-snaps Exercise

Reproduce the following screens on your TI-89.

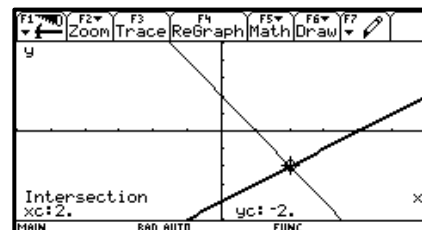
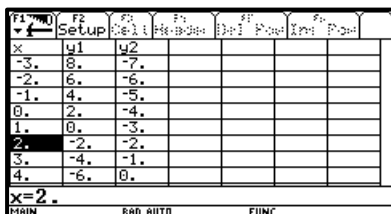
1.



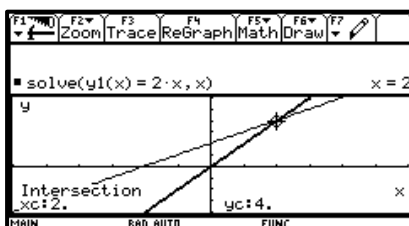
2.



3.



4.



For this question you will need to use the split-screen facility using:

[F2] 'Split Screen' – see page 3.

Investigation

Find all the integer values of a for which $ax + 1 = 3x + 5$ has integer solutions.

Inequalities

Example. Solve $3x - 2 = 7x + 10$

Method 1)

[F2] $3x - 2 \diamond \geq 7x + 10, x)$ [ENTER]

Note: The \geq automatically changes to \geq once the equation is entered.



Let us now solve the inequality step by step.

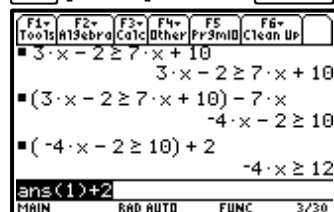
Method 2) In the following we transform in an equation into the form ' $x =$ or $= \dots$ ' by specifying equivalent transformations.

Step 1. $3x - 2 = 7x + 10$ [ENTER]

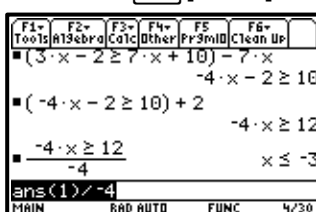
The subtraction of $7x$ is a reasonable first step.

Step 2. The application of the equivalent transformation of adding $-7x$ to both sides of the equation, adding 2 and dividing 4.

[2nd] [ANS] - 7x [ENTER]



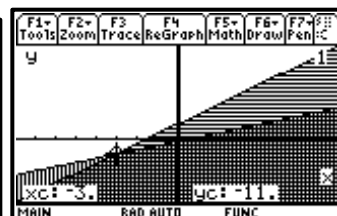
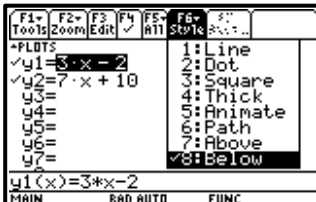
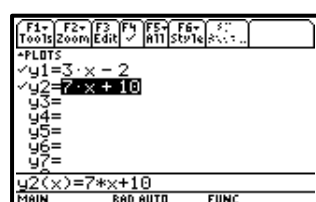
[2nd] [ANS] + 2 [ENTER]



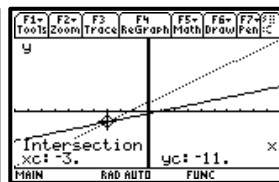
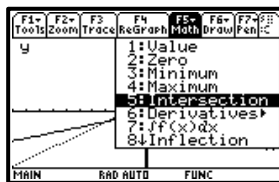
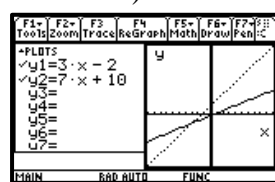
[2nd] [ANS] / -4 [ENTER]

Note: ans(1) always contains the last answer, ans(2), ans(3), etc, also contain previous answers. For example, ans(2) contains the next to last answer.

Method 3)



Method 4)



x	y1	y2
-5	-17	-25
-4	-14	-18
-3	-11	-11
-2	-8	-4
-1	-5	3

Exercise. Solve the following inequations:

1. $|4x - 2| = 6$

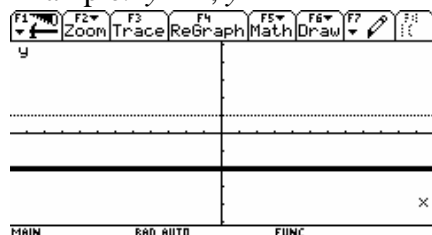
2. $|4x - 2| = 6$

5. Types of functions

Constant Function

$$f(x) = c$$

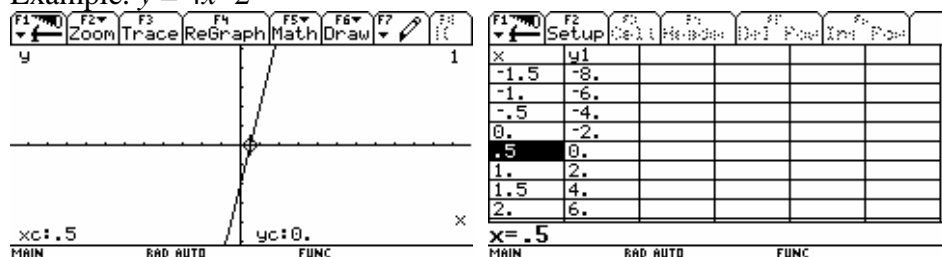
Example. $y = 1, y = -2$



Linear Function

$$f(x) = mx + b$$

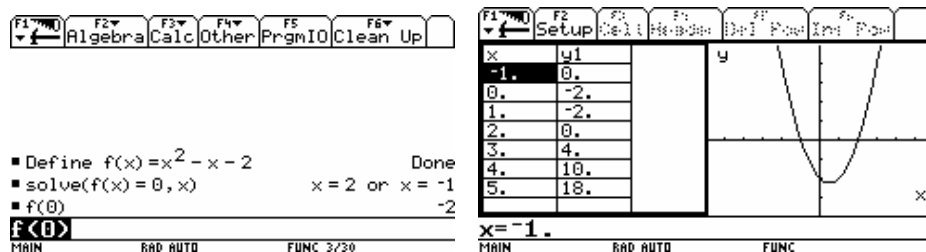
Example. $y = 4x - 2$



Quadratic Functions

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

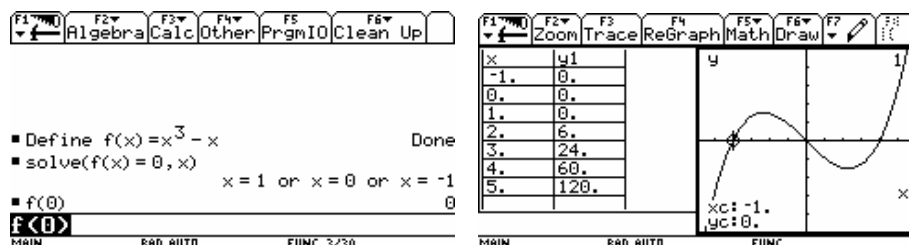
Example: $y = x^2 - x - 2 = (x - 2)(x + 1)$



Cubic Functions

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

Example. $y = x^3 - x = x(x + 1)(x - 1)$



Combinations of functions

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

Example. Let $f(x) = x^2 - x$, $g(x) = \frac{1}{x}$

$$(f + g)(x) =$$

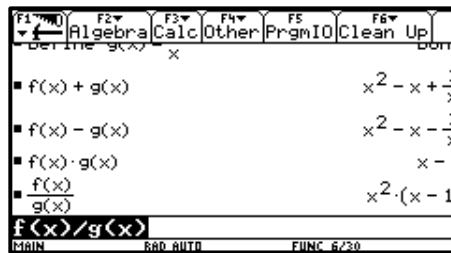
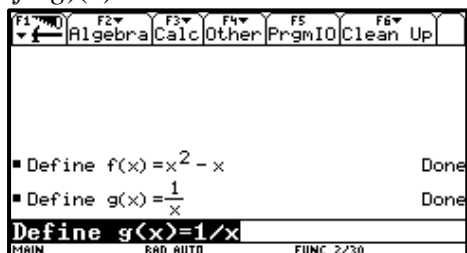
$$(f - g)(x) =$$

$$(f \bullet g)(x) = f(x) \bullet g(x)$$

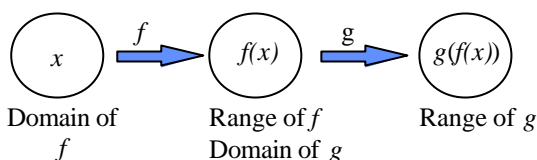
$$(f / g)(x) = f(x) / g(x)$$

$$(f \bullet g)(x) =$$

$$(f / g)(x) =$$



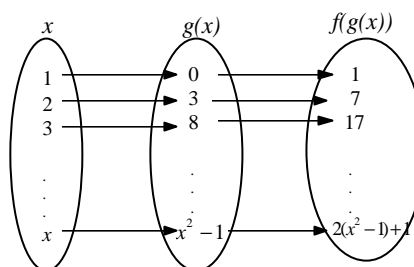
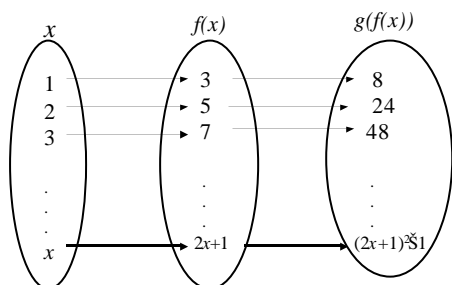
Composite Functions



Example. $f(x) = 2x + 1$ and $g(x) = x^2 - 1$

$$(g \circ f)(x) = (2x + 1)^2 - 1 = 4x(x + 1)$$

$$(f \circ g)(x) = 2(x^2 - 1) + 1 = 2x^2 - 1$$



Definition

$$(f \circ g)(x) = f(g(x))$$

cf. inverse $f \circ f^{-1} = I$

$$(g \circ f)(x) = g(f(x))$$

$f(g(x))$ is a function of a function. The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f .

Example. $f(x) = \sqrt{2x - 3}$, $g(x) = x^2 - 1$

$$(f \circ g)(x) =$$

$$(g \circ f)(x) =$$

$$(g \circ g)(x) =$$

The Exponential Function

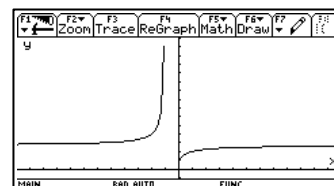
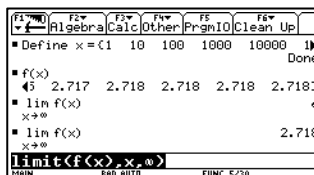
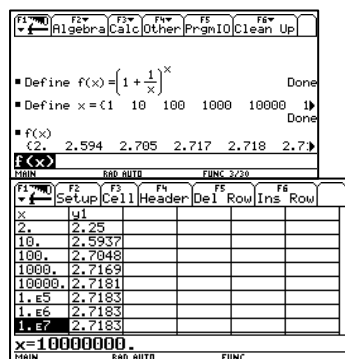
The exponential function is given by

$$f(x) = e^x$$

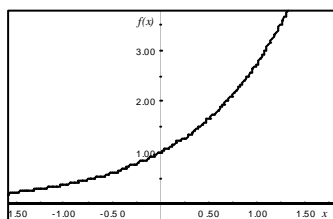
where the base “e” is approximately equal to 2.7182818284.

n	1	10	100	1000	10000	100000	1000000
$\left(1 + \frac{1}{n}\right)^n$	2	2.594	2.705	2.717	2.718	2.718	2.718

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828182845904... = e$$

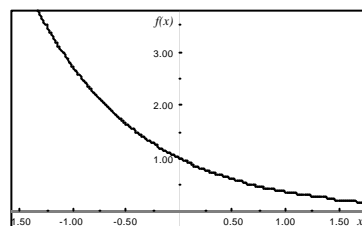


$$f(x) = e^x$$



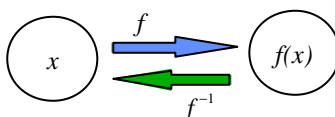
Domain: $x \in \mathbb{R}$
Range: $y > 0, y \in \mathbb{R}$

$$f(x) = e^{-x}$$



Domain: $x \in \mathbb{R}$
Range: $y > 0, y \in \mathbb{R}$

Inverse Functions



Definition.

Let f be a **one – to – one function** with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any y in B.

- Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

(The reciprocal $\frac{1}{f(x)}$ could be written as $[f(x)]^{-1}$)

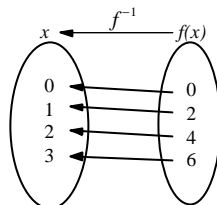
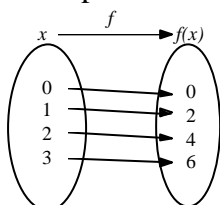
Example 1.

Find the inverse of the function f given by the following set:

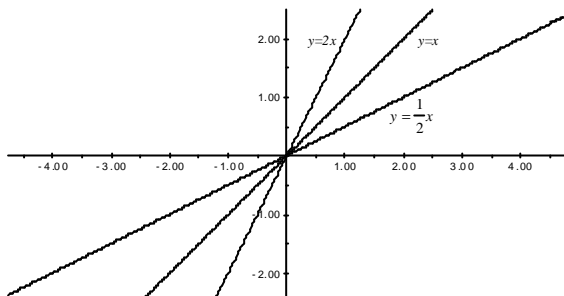
$$f = \{(10, 20) (15, 15) (25, 3) (27, 3)\}$$

$$\text{Answer: } f^{-1} = \{(20, 10) (15, 15) (3, 25) (3, 27)\}$$

Example 2. Find the inverse of the function $y = 2x$



If the function is given as a graph, you must reflect the graph in the line $y = x$ to find the graph of the inverse.



How to find the inverse function.

Step 1. Write $y = f(x)$.

Step 2. Solve this equation for x in terms of y .

Step 3. Interchange x and y . The resulting equation is $y = f^{-1}(x)$.

Example. Find the inverse function of $y = \sqrt{x}$.

Sol)

Step 1. Write $y = f(x)$.

$$y = \sqrt{x} \quad (x = 0, y = 0)$$

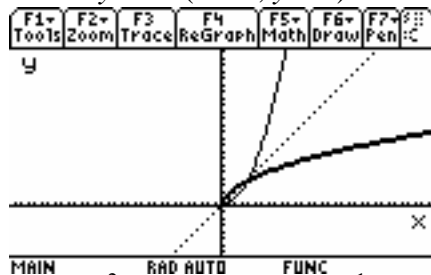
Step 2. Solve this equation for x in terms of y .

$$y^2 = x \quad (x = 0, y = 0)$$

$$\text{so } x = y^2$$

Step 3. Interchange x and y . The resulting equation is $y = f^{-1}(x)$.

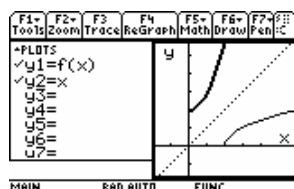
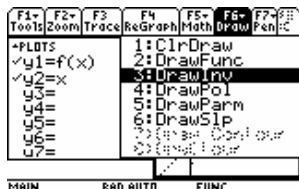
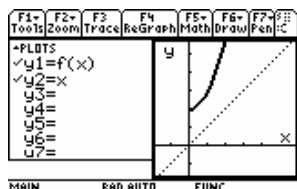
$$y = x^2 \quad (x = 0, y = 0)$$



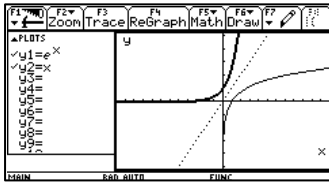
Example: For the given function, $f(x) = x^2 + 2$ ($x = 0$), find $f^{-1}(x)$, the inverse of f .

Solution: Since $x^2 = y - 2$, $x = \sqrt{y - 2}$ ($y = 2$)

The inverse function is $y = \sqrt{x - 2}$ ($x = 2$)



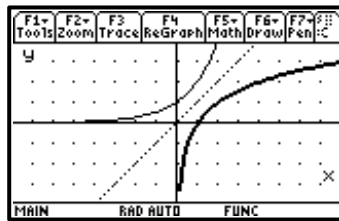
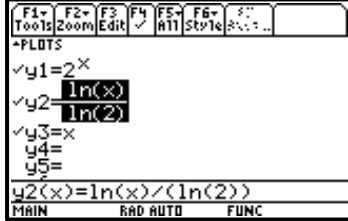
Example. $y = e^x$ and $y = \ln x$
(since $x = \ln y$)



Logarithmic & Exponential Functions

Logarithmic functions are the **inverse functions** to exponential functions.

Let $f(x) = 2^x$ and $f(x) = \log_2 x$ are a pair of inverse functions.



$$2^3 = 8 \Leftrightarrow 3 = \log_2 8$$

$$10^4 = 10000 \Leftrightarrow 4 = \log_{10} 10000$$

$$10^{0.4771} = 3 \Leftrightarrow 0.4771 = \log_{10} 3$$

$$3^n = 243 \Leftrightarrow n = \log_3 243$$

$$a^n = x \hat{=} n = \log_a x \quad (a > 0, a \neq 1)$$

(the logarithm of x to base a is said to be n)

Logarithms using base e are called natural logarithms, and $\log_e x = \ln x$

Rational functions

An asymptote is the *behaviour of a function* (or the graph of a function) *for extremely large values of x or y* . For *very large values of x or y* , graph of $y = f(x)$ gets close to the asymptote.

Rational functions are of the form : $f(x) = \frac{p(x)}{q(x)}$

(where $p(x)$ and $q(x)$ are polynomial expressions $q(x) \neq 0$)

Asymptotes:

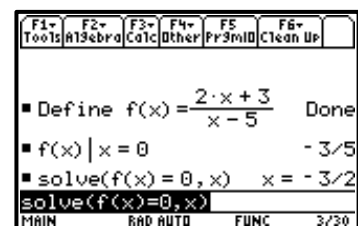
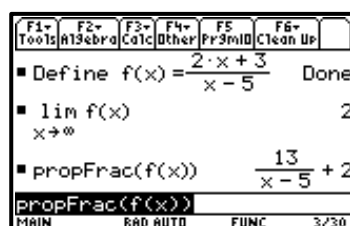
$$f(x) = \frac{ax+b}{cx+d} = \frac{k}{x-p} + q$$

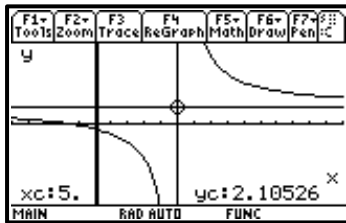
(To make simplify divide each term in the numerator and denominator by the highest power of x which appears)

- **Vertical asymptote:** $x = p$, (this is found by equating the *denominator to zero* and solving the resulting equation.)
- **Horizontal asymptote:** $y = q$, (this is found by finding the *limit of the function* as x gets very large.)
- Find the y -intercept by substituting $x = 0$ in the function.
- Find the x -intercept by equating $f(x) = 0$, and solving for x .
- Domain: $x \neq p, x \in \mathbb{R}$
- Range: $y \neq q, y \in \mathbb{R}$

Example. Sketch the function $f(x) = \frac{2x+3}{x-5}$, identifying all intercepts with the axes and all asymptotes.

$$f(x) = \frac{2x+3}{x-5} = \frac{13}{x-5} + 2$$





F1	F2	F3	F4	F5	F6	F7	F8
Tools	Setup	1/x	1/y	1/z	1/w	1/v	1/u
x	y1						
5.	undef						
6.	15.						
7.	8.5						
8.	6.3333						
9.	5.25						
x=9.							
MAIN	RAD AUTO	FUNC					

F1	F2	F3	F4	F5	F6	F7	F8
Tools	Setup	1/x	1/y	1/z	1/w	1/v	1/u
x	y1						
430.	2.0306						
431.	2.0305						
432.	2.0304						
433.	2.0304						
434.	2.0303						
x=434.							
MAIN	RAD AUTO	FUNC					

Vertical asymptote:

Horizontal asymptote:

y-intercept:

x-intercept:

Domain:

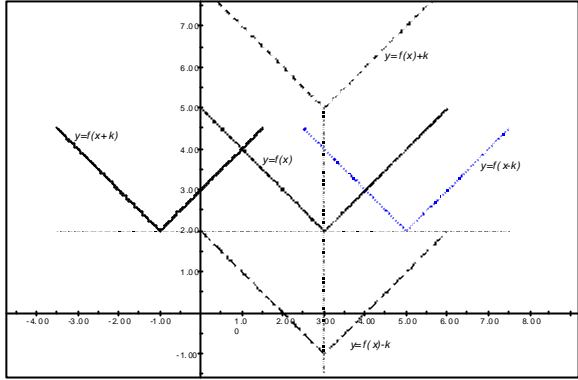
Range:

Exercise.

- Let $f(x) = \frac{x^2 - x - 6}{x + 2}$. Sketch the graph of $f(x)$ including any x and y intercepts. Can you explain why the graph has this form?
- Given $g(x) = \frac{2x + 3}{x - 5}$ is invertible on $x > 5$, find $f^{-1}(x)$, the inverse of f .

6. Transformations

$y = f(x) + k$	K units upward
$y = f(x) - k$	K units downward
$y = kf(x)$	Vertically by a factor of k
$y = -f(x)$	Reflect the graph of $y = f(x)$ in the x axis
$y = f(-x)$	Reflect the graph of $y = f(x)$ in the y axis
$y = f(x - m)$	Shift the graph of $y = f(x)$, m units to the right
$y = f(x + m)$	Shift the graph of $y = f(x)$, m units to the left



Transformations parallel to the x - and y -axis

Example. The purpose here is to explain the relationship between $f(x)$, $f(x-a)$ and $f(x)+b$. Define the function $f(x)=x^2$.

If $f(x)$ is defined by x^2 then $f(x-a)$ and $f(x)+b$ are found to be $(x-a)^2$ and x^2+b by following these instructions:

[HOME] [F4] 1 $f(x)=x^2$ [ENTER]

$f(x-a)$ [ENTER]

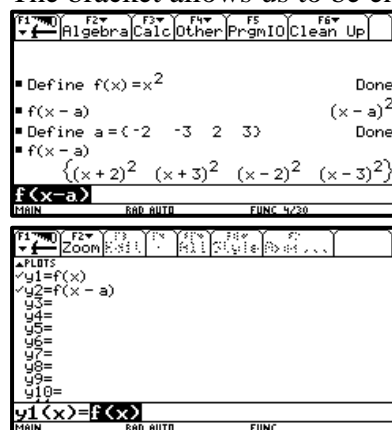
[F4] 1 $a = \{-2, -3, 2, 3\}$ [ENTER]

[HOME] [F4] 1 $f(x)=x^2$ [ENTER]

$f(x)+b$ [ENTER]

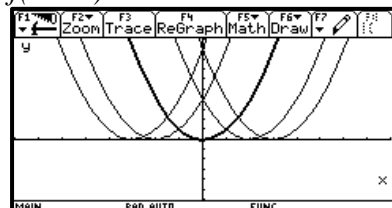
[F4] 1 $b = \{-2, -3, 2, 3\}$ [ENTER]

The bracket allows us to be enter a number of different values for a or b .

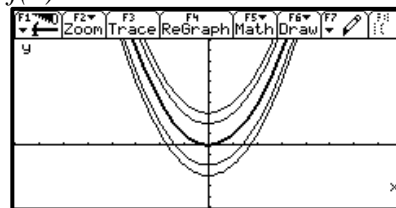


Note that when drawing $f(x-a)$ and $f(x)+b$ the calculator uses each of the values of a or b entered, showing the effect of them. We can see that the effect of

$f(x-a)$



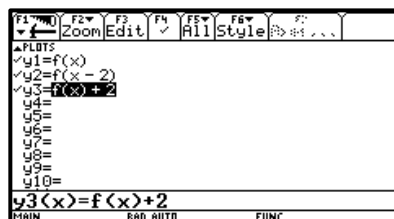
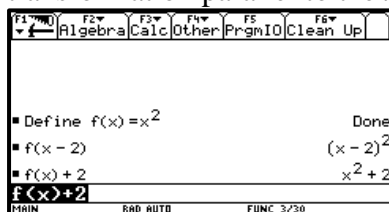
$f(x)+b$



We can also deal with single values.

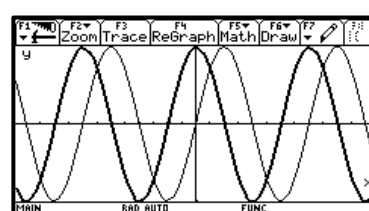
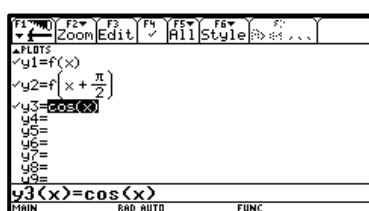
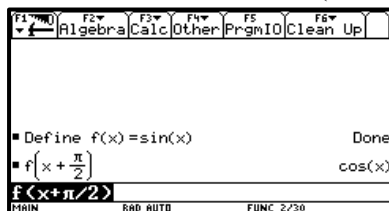
e.g. Compare the general functions $f(x-2)$ and $f(x)+2$ for $f(x)=x^2$.

Looking at the general functions $f(x-2)$ and $f(x) + 2$ we can see that those functions correspond to the actual functions $(x-2)^2$ and $x^2 + 2$ based on the function $f(x) = x^2$, and the graph shows the transformation parallel to the x - and y -axes.

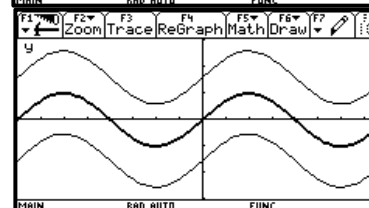
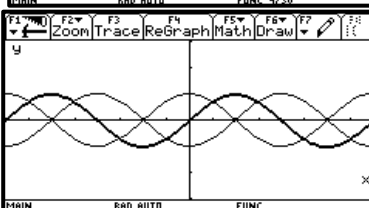
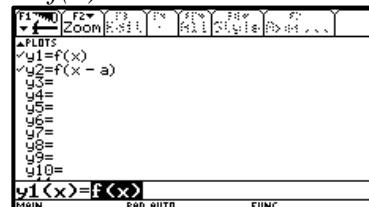
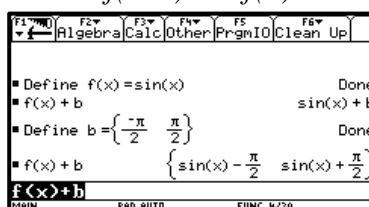
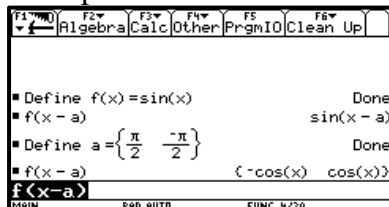


Sine and Cosine Function

Example. Show that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$



Example. Find the difference between $f(x-a)$ and $f(x) + b$ when $f(x) = \sin x$.



7. Limits

• The Calc Menu

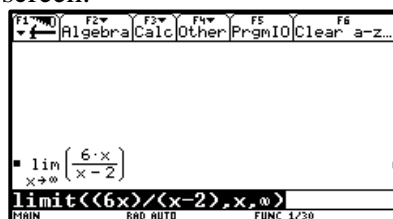
From the Home screen, press [F3].

Menu Item	Description
d differentiate	Differentiates an expression with respect to a specified variable
\int integrate	Integrates an expression with respect to a specified variable.
limit	Calculates the limit of an expression with respect to a specified variable

To find $\lim_{x \rightarrow \infty} \frac{6x}{x-2}$ follow the key sequence.

[F3] 3 (6 x) / (x - 2) , x , \blacklozenge [CATALOG]) [ENTER]

The following should appear on your calculator screen.



Note: Put both numerator and denominator in brackets.

Example 1. Find $\lim_{x \rightarrow 0} x \cos x$

We can get a sense of the limit by defining the function as $f(x)$ and getting values of x near to zero.

To find $\lim_{x \rightarrow 0} x \cos x$ follow the key sequence:

[F4] 1 $f(x) = x \cos(x)$ [ENTER]

[F3] 3 $f(x)$, x , 0) [ENTER]

Whenever we change x taking steps of x closer to 0 then the value of $f(x)$ is getting closer to 0.

We can confirm our guess by asking for the limit.

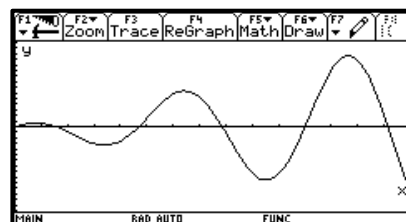
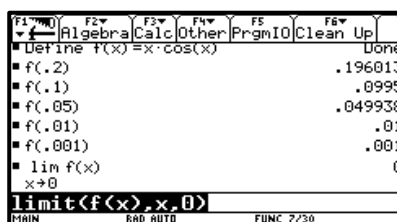
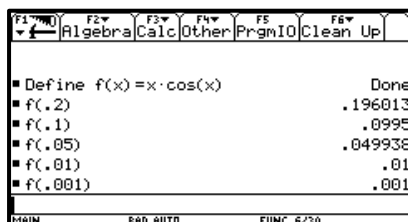


TABLE SETUP

tblStart: -1.2

tbl: 1.2

Graph <-> Table: OFF+

Independent: AUTO+

Enter=SAVE ESC=CANCEL

x=0.

x	y1
-1.2	-.196
-1	0.
-.2	.19601
.4	.36842
.6	.4952
.8	.55737
1.	.5493
1.2	.43483

x=0.

x	y1
-1.2	-.196
-.1	-.0995
0.	0.
.1	.0995
.2	.19601
.3	.2866
.4	.36842
.5	.43879

x=0.

x	y1
0.	0.
.01	.01
.02	.02
.03	.02999
.04	.03997
.05	.04994
.06	.05989
.07	.06983

x=0.

x	y1
0.	0.
.001	.001
.002	.002
.003	.003
.004	.004
.005	.005
.006	.006
.007	.007

x=0.

The graph and table help to confirm, in other representations, that the function has a limit of zero when $x \rightarrow 0$.

Example 2. Find $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

This is an important limit, but one that cannot be found by putting $x = 0$, since the function is undefined for $x = 0$.

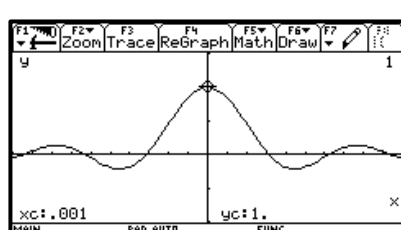
[F4] 1 $f(x) = \sin(x) \div x$) [ENTER]

Whenever we change x taking steps of x closer to 0 then the value of $f(x)$ is getting closer to 1.

[F3] 3 $f(x), x, 0$) [ENTER]

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
Define f(x) = $\frac{\sin(x)}{x}$ Done					
f(.2)				.993347	
f(.1)				.998334	
f(.05)				.999583	
f(.01)				.999983	
f(.001)				1.	
f(0.001)					

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
f(.2) .993347					
f(.1) .998334					
f(.05) .999583					
f(.01) .999983					
f(.001) 1.					
lim f(x)					
x → 0					
limit(f(x), x, 0)					



F1	F2	F3	F4	F5	F6
Setup	Cell	Header	Def	Row	Col
x	y1				
0.	undef				
.2	.99335				
.4	.97355				
.6	.94107				
.8	.8967				
1.	.84147				
1.2	.7767				
1.4	.70389				
x=0.					

F1	F2	F3	F4	F5	F6
Setup	Cell	Header	Def	Row	Col
x	y1				
0.	undef				
.1	.99833				
.2	.99335				
.3	.98507				
.4	.97355				
.5	.95885				
.6	.94107				
.7	.92031				
x=0.					

F1	F2	F3	F4	F5	F6
Setup	Cell	Header	Def	Row	Col
x	y1				
0.	undef				
.01	.99998				
.02	.99993				
.03	.99985				
.04	.99973				
.05	.99958				
.06	.9994				
.07	.99918				
x=0.					

F1	F2	F3	F4	F5	F6
Setup	Cell	Header	Def	Row	Col
x	y1				
0.	undef				
.001	1.				
.002	1.				
.003	1.				
.004	1.				
.005	1.				
.006	.99999				
.007	.99999				
x=0.					

Again the graph and table provide supporting evidence for the limit.

Example 3. $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$

[F4] 1 $f(x) = x^2 \sin(x)$) [ENTER]

[F3] 3 $f(x), x, 0$) [ENTER]

♦ [y=]

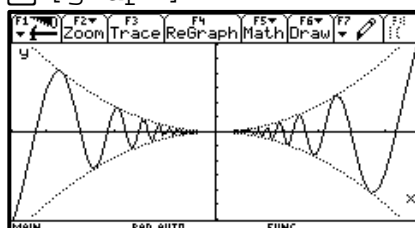
F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
Define f(x) = $x^2 \cdot \sin\left(\frac{1}{x}\right)$ Done					
lim f(x)					
x → 0					
limit(f(x), x, 0)					

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrgmIO	Clean Up	
y1 = $x^2 \cdot \sin\left(\frac{1}{x}\right)$					
y2 = x^2					
y3 = $-x^2$					
y4 =					
y5 =					
y6 =					
y7 =					
y8 =					
y3(x) = $-x^2$					

♦ [window]

F1	F2
Zoom	
xmin = -1.2	
xmax = .12	
xsc1 = .02	
ymin = -.012	
ymax = .012	
ysc1 = .002	
xres = 2.	

♦ [graph]



TABLE

F1	F2	F3	F4	F5	F6	F7	F8
Setup	Cell	Header	Del	Pow	Ins	Row	Col
X	Y1	Y2	Y3	Y4	Y5	Y6	Y7
0.	0.	0.	0.	0.	0.	0.	0.
.5	.22732	.25	-.25				
1.	.84147	1.	-1.				
1.5	1.3913	2.25	-2.25				
2.	1.9177	4.	-4.				
2.5	2.4339	6.25	-6.25				
3.	2.9448	9.	-9.				
3.5	3.4526	12.25	-12.25				
x=0.							
MAIN RAD AUTO FUNC							

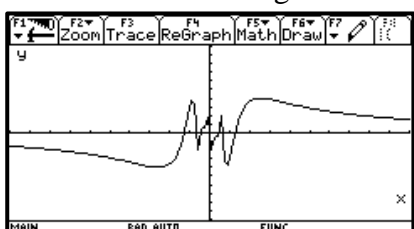
F1	F2	F3	F4	F5	F6	F7	F8
Setup	Cell	Header	Del	Pow	Ins	Row	Col
X	Y1	Y2	Y3	Y4	Y5	Y6	Y7
0.	0.	0.	0.	0.	0.	0.	0.
.1	-.0054	.01	-.01				
.2	-.0384	.04	-.04				
.3	-.0817	.09	-.09				
.4	-.0957	.16	-.16				
.5	-.2273	.25	-.25				
.6	-.3583	.36	-.36				
.7	-.4856	.49	-.49				
x=0.							
MAIN RAD AUTO FUNC							

F1	F2	F3	F4	F5	F6	F7	F8
Setup	Cell	Header	Del	Pow	Ins	Row	Col
X	Y1	Y2	Y3	Y4	Y5	Y6	Y7
0.	0.	0.	0.	0.	0.	0.	0.
.001	8.3E-7	1.E-6	-1.E-6				
.002	-2.E-6	4.E-6	-4.E-6				
.003	2.9E-6	9.E-6	-9.E-6				
.004	-2.E-5	.00002	-2.E-5				
.005	-2.E-5	.00003	-3.E-5				
.006	-6.E-6	.00004	-4.E-5				
.007	-5.E-5	.00005	-5.E-5				
x=0.							
MAIN RAD AUTO FUNC							

Example 4. $\lim_{x \rightarrow 0} \sin \frac{1}{x}$

Some limits do not exist. We can build an understanding of the reasons for this.

F1	F2	F3	F4	F5	F6	F7	F8
Algebra	Calc	Other	PrgmIO	Clean Up			
Define f(x)=sin(1/x)							
lim f(x) x→0 undef							
limit(f(x),x,0)							
MAIN RAD AUTO FUNC 2/30							



F1	F2	F3	F4	F5	F6	F7	F8
Setup	Cell	Header	Del	Pow	Ins	Row	Col
X	Y1	Y2	Y3	Y4	Y5	Y6	Y7
-1	.54402						
0.	sin(x)						
.1	-.544						
.2	-.9589						
.3	-.1906						
.4	.59847						
.5	.9093						
.6	.99541						
x=0.							
MAIN RAD AUTO FUNC							

F1	F2	F3	F4	F5	F6	F7	F8
Setup	Cell	Header	Del	Pow	Ins	Row	Col
X	Y1	Y2	Y3	Y4	Y5	Y6	Y7
-1	.54402						
-.05	-.9129						
0.	sin(x)						
.05	.91295						
.1	-.544						
.15	.37415						
.2	-.9589						
.25	-.7568						
x=-.1							
MAIN RAD AUTO FUNC							

F1	F2	F3	F4	F5	F6	F7	F8
Setup	Cell	Header	Del	Pow	Ins	Row	Col
X	Y1	Y2	Y3	Y4	Y5	Y6	Y7
-.03	-.9405						
-.02	.26237						
-.01	.50637						
0.	sin(x)						
.01	-.5064						
.02	-.2624						
.03	.94053						
.04	-.1324						
x=0.							
MAIN RAD AUTO FUNC							

We can plot the graph and zoom in on $x = 0$ or from the table we can see that no matter how much we zoom in on $x = 0$ values either side are the same but differ in sign. This leads to the idea of left and right limits.

Left and Right Limits and Differential Functions

We can use the left and right limits to see why some functions are not differentiable at certain points.

Consider the expression

$$f(t) = \frac{t^2 - 7t + 10}{t - 2}$$

Define the function: [F4] 1 $f(t) = (t^2 - 7t + 10) \div (t - 2)$ [ENTER]

Investigate right limit: [F4] 1 $t = \{1.9, 1.99, 1.999, 1.9999\}$ then evaluate $f(t)$

Investigate left limit: [F4] 1 $t = \{2.1, 2.01, 2.001, 2.0001\}$ then evaluate $f(t)$

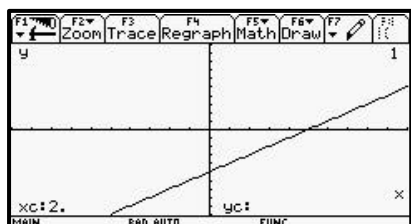
Right limit is: [F3] 3 $f(t), t, 2, -1$ [ENTER]

Left limit is: [F3] 3 $f(t), t, 2, 1$ [ENTER]

F1	F2	F3	F4	F5	F6	F7	F8
Algebra	Calc	Other	PrgmIO	Clean Up			
Define f(t)=(t^2-7t+10)/(t-2)							
Define t={1.9 1.99 1.999 1.9999}							
f(t) {-3.1 -3.01 -3.001 -3.0001}							
Define t={2.1 2.01 2.001 2.0001}							
f(t) {2.1 2.01 2.001 2.0001}							
ne t={2.1,2.01,2.001,2.0001}							
MAIN RAD AUTO FUNC 4/30							

F1	F2	F3	F4	F5	F6	F7	F8
Algebra	Calc	Other	PrgmIO	Clean Up			
f(t) {-3.1 -3.01 -3.001 -3.0001}							
Define t={2.1 2.01 2.001 2.0001}							
lim f(t) t→2+							
lim f(t) t→2-							
limit(f(t),t,2,-1)							
MAIN RAD AUTO FUNC 6/30							

F1	F2	F3	F4	F5	F6	F7	F8
Zoom	Edit	Full	Style	Fit	Style	Fit	Style
y1=x^2-7x+10							
y2=x-2							
y3=							
y4=							
y5=							
y6=							
y7=							
y8=							
y9=							
y2(x)=							
MAIN RAD AUTO FUNC							



F1	F2	F3	F4	F5	F6	F7	F8
Setup	Cell	Header	Del	Row	Ins	Row	Col
X	Y1	Y2	Y3	Y4	Y5	Y6	Y7
1.9	-3.1						
1.99	-3.01						
1.999	-3.001						
2.	undef						
2.1	-2.9						
2.01	-2.99						
2.001	-2.999						
2.0001	-2.9999						
x=2.0001							
MAIN RAD AUTO FUNC							

Example 5. Find $\lim_{x \rightarrow 2} f(x)$ for the function $f(x) = \begin{cases} x^2 & \text{for } x < 2 \\ 6 - x & \text{for } x \geq 2 \end{cases}$.

Define the piecewise functions by using the following instructions.

[F4] 1 $f(x) = \text{when } (x^2, x < 2, 6 - x)$ [ENTER]

Investigate right limit: [F4] 1 $x = \{1.9, 1.99, 1.999, 1.9999\}$ then evaluate $f(x)$

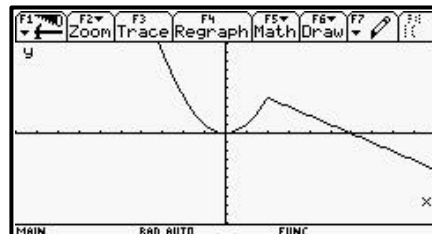
Investigate left limit: [F4] 1 $x = \{2.1, 2.01, 2.001, 2.0001\}$ then evaluate $f(x)$

Right limit is: [F3] 3 $f(x), x, 2, -1$) [ENTER]

Left limit is: [F3] 3 $f(x), x, 2, 1$) [ENTER]

The image shows a TI-89 calculator screen. The top part shows the function definition: $f(x) = \begin{cases} x^2, & x < 2 \\ 6 - x, & \text{else} \end{cases}$. Below this, a table of values is shown for x values approaching 2 from both sides. The bottom part of the screen shows a table with columns for x and $f(x)$.

x	$f(x)$
1.9	3.61
1.99	3.9601
1.999	3.996001
2.	4.
2.1	3.9
2.01	3.99
2.001	3.999
2.0001	3.9999



Exercise.

Using the symbolic, graphical and tabular representations find these limits if possible.

- $\lim_{x \rightarrow 2} (3x - 1)$
- $\lim_{x \rightarrow 2} \frac{x^2 + 5x - 14}{x^2 - x - 2}$
- $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$
- $f(x) = \begin{cases} x & (x \leq 0) \\ x^2 - 2x - 3 & (x \geq 0) \end{cases}, \lim_{x \rightarrow 0} f(x)$
- $\lim_{x \rightarrow 0} |x|$
- $\lim_{x \rightarrow -3^-} \frac{\sqrt{x^2 - 9}}{x + 3}$

Techniques for finding limits

- Numerically (substitute numbers from both sides)
- Direct substitution
- Algebraic Cancellation then substitution
- Limits as $x \rightarrow \infty$ (divide top and bottom by the highest power of x)

Summary table for common cases if you substitute first:

Result when substituting	Conclusion
Sensible answer	This is the limit
$\frac{\text{number} \neq 0}{0}$	Limit does not exist
$\frac{0}{\text{number} \neq 0}$	Limit = 0
$\frac{0}{0}$	Factorise, cancel, and try again

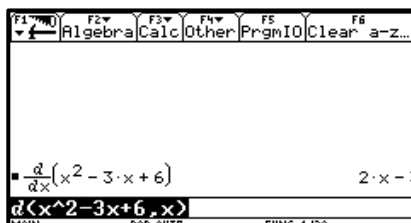
8. Differentiation

We can get the calculator to differentiate directly and give the answer:

To differentiate the function $y = x^2 - 3x + 6$ follow these key sequence instructions:

[F3] 1 $x^2 - 3x + 6, x$) [ENTER]

The following should appear on the calculator display.



Notes:

We type **comma x** at the end of the expression because we are differentiating with respect to x .

There is no need to type in the multiplication sign between 3 and x .

All expressions are enclosed in brackets.

Exercise

Find the derivative of each of the following functions using the TI-89

- | | | |
|----------------------------|-------------------------------|-----------------------------------|
| 1. $x^2 + 5x^3$ | 2. $20x^8 + 9x^3 + 52$ | 3. $(x-6)(x+5)$ |
| 4. $\frac{x^2 - 9}{x + 3}$ | 5. $\frac{e^{-2x}}{3e^x - 1}$ | 6. $2x^3 \sin^2 x - \cos(2x - 1)$ |
| 7. $3x^2 \ln x$ | 8. $\frac{3}{x^2}$ | 9. $\sqrt{9x^2 - 36}$ |

Answers:

- | | | | |
|--|---|--------------------------------|------|
| 1. $15x^2 + 2x$ | 2. $27x^2 + 160x^7$ | 3. $2x - 1$ | 4. 1 |
| 5. $\frac{-(9e^x - 2)e^{-2x}}{(3e^x - 1)^2}$ | 6. $2 \sin(2x - 1) + 4x^3 \sin(x) \cos(x) + 6x^2 (\sin(x))^2$ | | |
| 7. $6x \ln(x) + 3x$ | 8. $\frac{-6}{x^3}$ | 9. $\frac{3x}{\sqrt{x^2 - 4}}$ | |

Example 1. Find the derivative of $f(x) = x^2$ at $x = 2$.

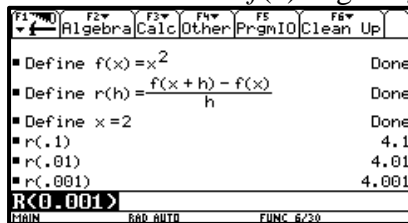
We can do differentiation from first principles by using the ideas of limits we have developed.

Method 1.

[F4] 1 $f(x) = x^2$ [ENTER]

In this method we use the calculator function $r(h)$ (i.e. rate of change) at the point $x = 2$:

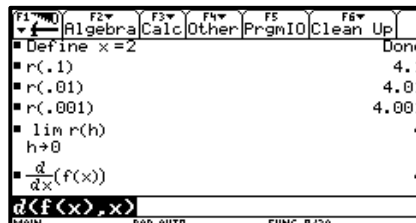
Whenever we change h taking steps of h closer to 0 then the value of $f(x)$ is getting closer to 4.



[F3] 3 $r(h), h, h, 0) [] x = 2$ [ENTER]

[F3] 1 $f(x), x) [] x = 2$ [ENTER]

We can confirm our guess by asking for the limit and differentiation.

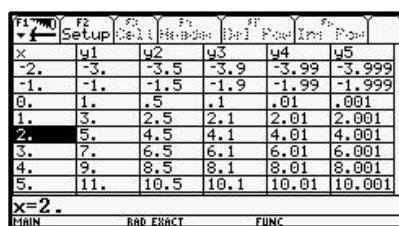
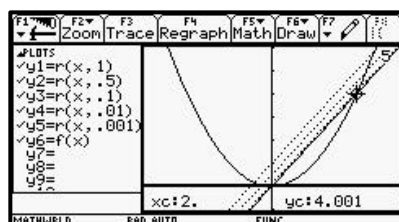
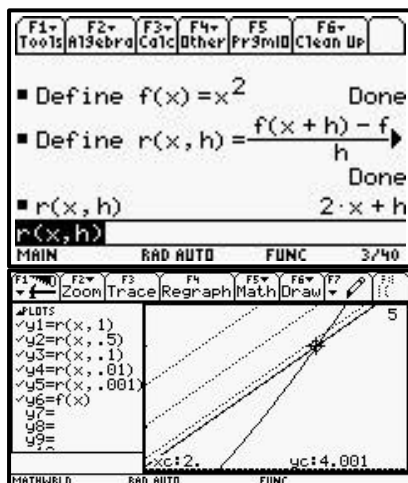


Thus the rate of change at $x = 2$: $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = f'(2) = 4$.

Method 2.

In this method we use the calculator function $r(x, h)$ (i.e. rate of change) at the point $x = 2$:

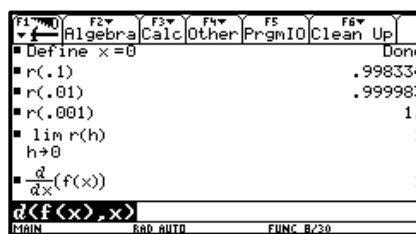
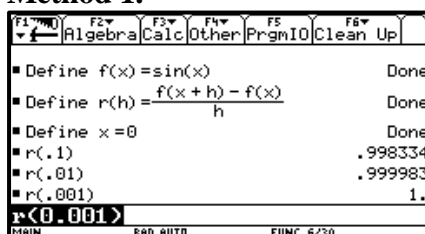
Whenever we change h taking steps of h closer to 0 then the value of $f(x)$ is getting closer to 4.



We can see the general function $2x$.

Example 2. Find the derivative of $f(x) = \sin x$

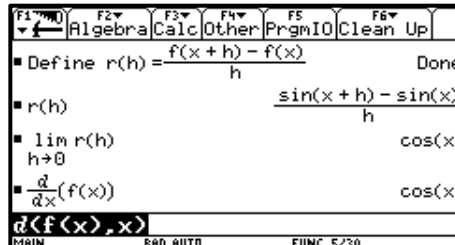
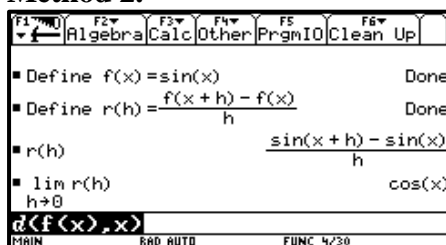
Method 1.



$$\frac{dy}{dx} = f'(0) = \lim_{h \rightarrow 0} \frac{\sin(0+h) - \sin(0)}{h} = \cos(0) = 1$$

$$\frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} = \cos(x)$$

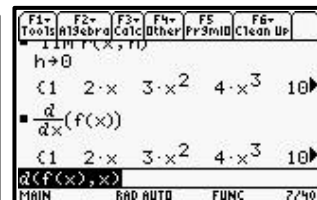
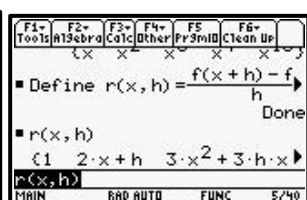
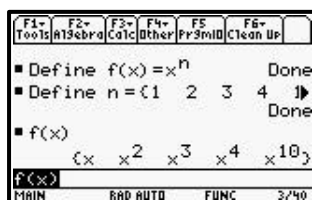
Method 2.



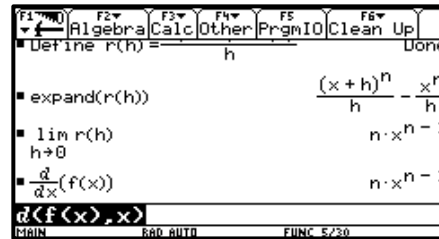
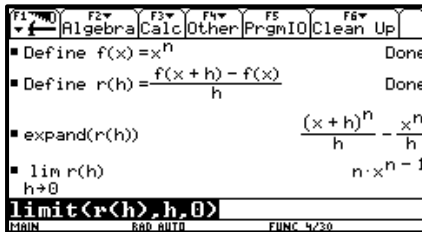
Example. Find the derivative of $f(x) = x^n$

This example can be difficult from first principles if students do not have access to the binomial theorem.

Define the function $f(x) = x^n$. When we define the value of power, $n = 1, 2, 3, 4, 10$ the functions are changed to the actual functions, x, x^2, x^3, x^4, x^{10} . If we define the slope function $slope(h)$ as the average rate of changed, then we can see that the derivative of the functions are $1, 2x, 3x^2, 4x^3, 10x^9$ as follows:



Defining the rate of function $r(h)$, we can get that the general derivative of x^n is nx^{n-1} as follows:



Thus $\frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}$

Differentiation Formulas

$f(x)$	$f'(x)$
a) $f(x) = c$ (c is constant)	$f'(x) = 0$
b) $y = x^n$	$y' = nx^{n-1}$
c) $y = c \cdot f(x)$ (c is constant)	$y' = c \cdot f'(x)$
d) $y = f(x) + g(x)$	$y' = f'(x) + g'(x)$

Product rule

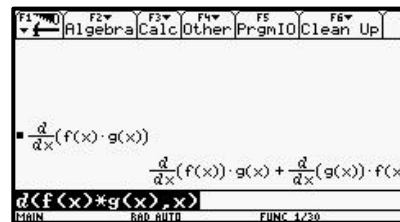
Where $y = u \cdot v$ and u and v are both functions of x , then:

$$\frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

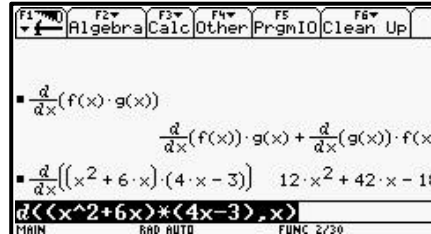
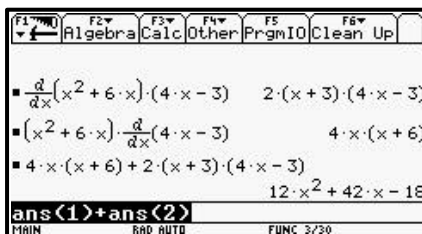
or

$$y = f(x)g(x)$$

$$y' = f'(x)g(x) + f(x)g'(x)$$



Example. Find the derivative of the function $y = (x^2 + 6x)(4x - 3)$



Exercise. Differentiate the following using the product rule.

1. $y = (x^3 - 3)(x^2 + 2)$

2. $y = \frac{2x^2 + 1}{x^2}$

3. $y = x^3 \sqrt{x}$

4. $y = \sqrt{x}(3x^2 - 1)$

Quotient rule

Where $y = \frac{u}{v}$ and u and v are both functions of x then

$$\frac{dy}{dx} = \frac{\frac{du}{dx} \cdot v - u \cdot \frac{dv}{dx}}{v^2}$$

or

$$y = \frac{f(x)}{g(x)} \quad (g(x) \neq 0)$$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}$$

pf)

TI-89 calculator screen showing the quotient rule formula for differentiation. The screen displays the formula:
$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}$$

TI-89 calculator screen showing the quotient rule formula for differentiation. The screen displays the formula:
$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}$$

$$y = \frac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}$$

$$y' = f'(x) \cdot g(x)^{-1} - f(x) \cdot g(x)^{-2} \cdot g'(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x) \cdot g'(x)}{g(x)^2}$$

$$= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Example. If $y = \frac{x^2}{3x-2}$

TI-89 calculator screen showing the differentiation of $y = \frac{x^2}{3x-2}$. The screen displays the steps: Define $f(x) = x^2$, Define $g(x) = 3x-2$, and then the derivative calculation:
$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{2x \cdot (3x-2) - x^2 \cdot 3}{(3x-2)^2}$$

TI-89 calculator screen showing the differentiation of $y = \frac{x^2}{3x-2}$. The screen displays the steps: Define $f(x) = x^2$, Define $g(x) = 3x-2$, and then the derivative calculation:
$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{2x \cdot (3x-2) - x^2 \cdot 3}{(3x-2)^2}$$

Exercise. Differentiate the following using the quotient rule.

1. $y = \frac{x-1}{x+3}$

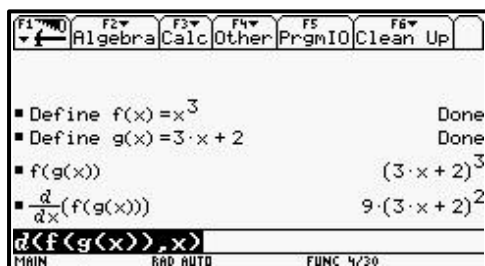
2. $y = \frac{x^2-6x}{x-3}$

3. $y = \frac{x-x^3}{\sqrt{x}}$

Chain Rule

Composite Function	
If $y = f(u)$, $u = g(x)$	
then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ or $f'(g(x)) \cdot g'(x)$	
$y = f(ax+b)$	$\frac{dy}{dx} = af'(ax+b)$
$y = \{f(x)\}^n$	$\frac{dy}{dx} = n\{f(x)\}^{n-1} f'(x)$
<p>TI-89 calculator screen showing the power rule for differentiation. The screen displays the formula: $\frac{d}{dx} \left((f(x))^n \right) = (f(x))^{n-1} \cdot \frac{d}{dx}(f(x)) \cdot n$</p>	

Example. If $y = (3x + 2)^3$ find $\frac{dy}{dx}$



Exercise. Differentiate each of these.

1. $y = (7x + 5)^5$

2. $y = (4x^2 + 2x)^3$

3. $y = (7x - x^2)^{-2}$

4. $y = (ax + b)^3$

5. $y = \left(\frac{3}{5}x - \frac{1}{2}\right)^2$

6. $y = \frac{4}{\sqrt{x+3}}$

7. $y = \frac{1}{1 + \sqrt{x}}$

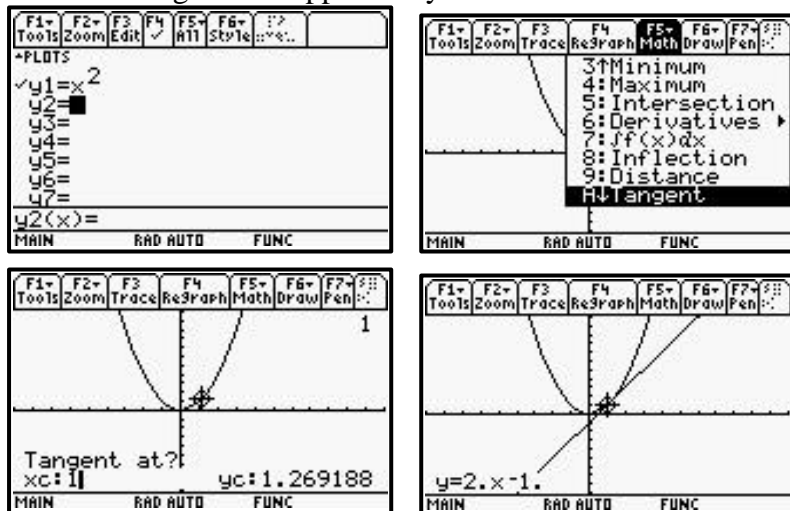
8. $y = (x + \sqrt{x})^4$

9. Finding the tangent line at a point on curve

To Find the equation of the tangent to $y = x^2$ at $x=1$

◊ [y=] x^2 [ENTER] ◊ [Graph] [F2] - 4 [F5] **A:Tangent**

The following should appear on your calculator:



Exercise.

Find the tangent line to a curve.

1. $2x^2 - x - 15$, at $x = -1$

2. $2x^3 - 4x^2 - 6x$, at $x = -2$

3. $2x^4 - 6x^3 - 2x^2 + 6x$, at $x = 3$

4. $-2x^4 + x^3 + 17x^2 - x - 15$, at $x = 3$

Answers:

1. $y = -5x - 17$

2. $y = 34x + 48$

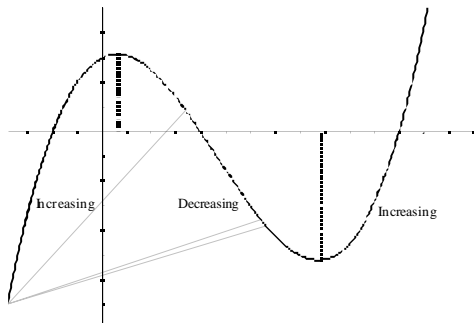
3. $y = 48x - 144$

4. $y = -88x + 264$

The Increasing/Decreasing Concept

The increasing/decreasing concept can be associated with the slope of the tangent line.

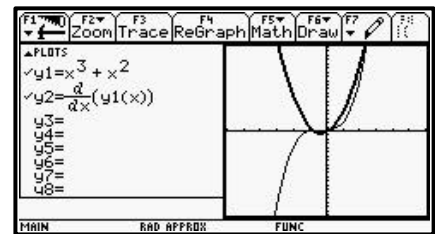
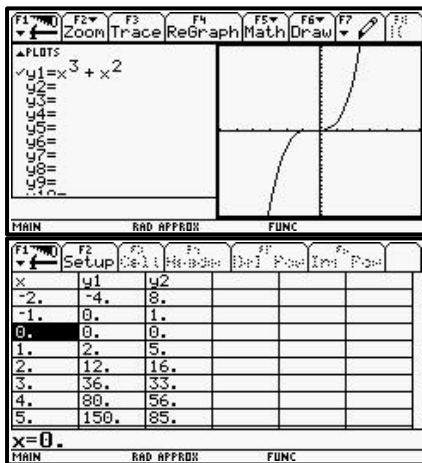
1. At a point (at which f is defined)
 - (j) If $f'(a) > 0$, then f is increasing at $x = a$
 - (k) If $f'(a) < 0$, then f is decreasing at $x = a$
2. On an interval (on which f is defined)
 1. If $f'(a) > 0$ for all x in an interval, then f is increasing on the interval.



2. If $f'(a) < 0$ for all x in an interval, then f is decreasing on the interval.

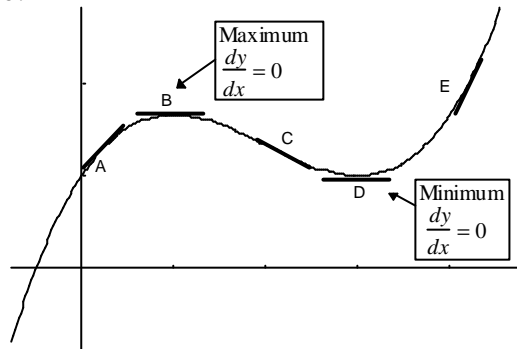
Example. If $f(x) = x^3 + x^2$, is increasing or decreasing at $x = 5$?

Find the intervals on which $f(x)$ is increasing or decreasing?



First Derivative Test

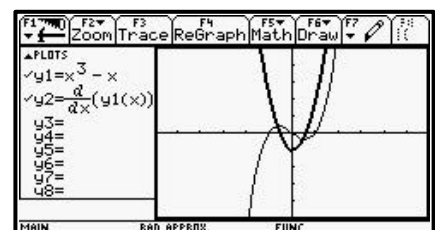
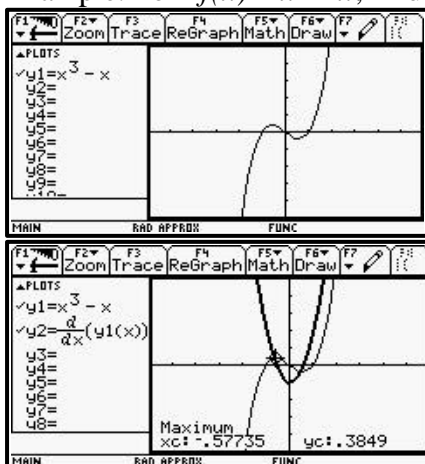
- Let c be a critical number of f and let f be continuous on an interval containing c . Then $(c, f(c))$ is a *relative maximum point* provided that $f'(x) > 0$ is an interval to the left of c and $f'(x) < 0$ in an interval to the right of c .
- Let c be a critical number of f and let f be continuous on an interval containing c . Then $(c, f(c))$ is a *relative minimum point* provided that $f'(x) < 0$ is an interval to the left of c and $f'(x) > 0$ in an interval to the right of c .

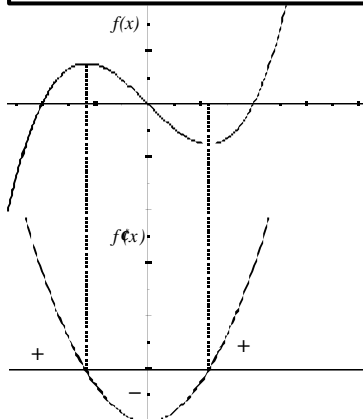
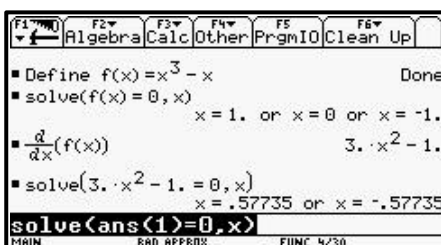
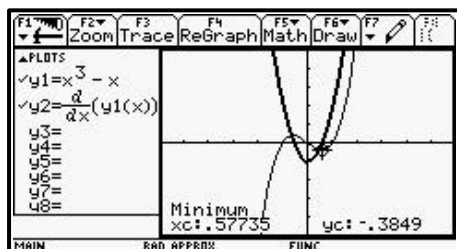


To find Maximum & Minimum Values:

- Find critical points ($f'(x)=0$) \rightarrow x value
- Substitute into $f(x)$
 If $f'(x)$ changes $+$ \rightarrow $-$ Maximum value
 $-$ \rightarrow $+$ Minimum value

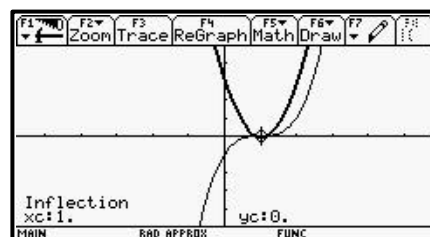
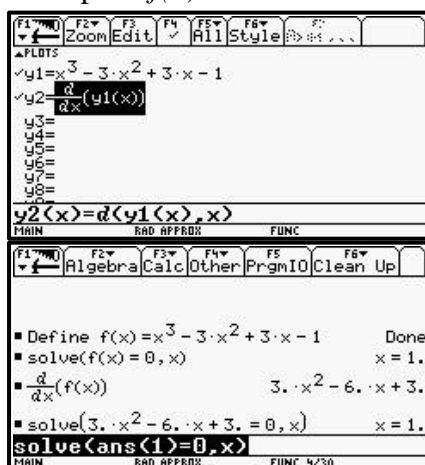
Example. For $f(x) = x^3 - x$, find maximum and minimum values.





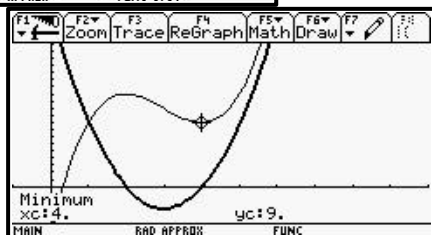
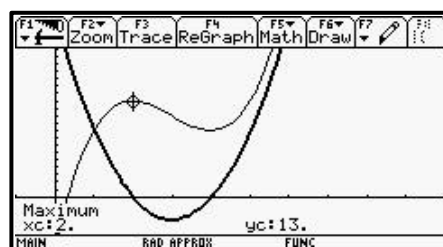
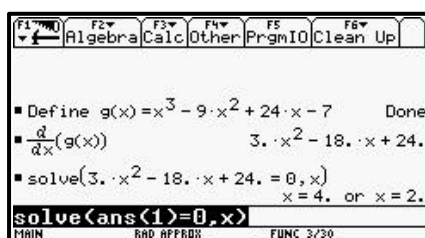
(e) The derivative can be zero without there being a relative maximum or relative minimum.

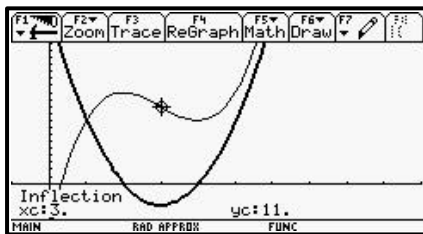
Example. $f(x) = x^3 - 3x^2 + 3x - 1$



Local Maxima and Minima

Example. Find all local maxima and minima of the function $g(x) = x^3 - 9x^2 + 24x - 7$ and sketch graph.





F1	F2	F3	F4	F5	F6	F7	F8
Setup	Del	2nd	3rd	4th	5th	6th	7th
8th	9th	10th	11th	12th	13th	14th	15th
16th	17th	18th	19th	20th	21st	22nd	23rd
24th	25th	26th	27th	28th	29th	30th	31st
32nd	33rd	34th	35th	36th	37th	38th	39th
40th	41st	42nd	43rd	44th	45th	46th	47th
48th	49th	50th	51st	52nd	53rd	54th	55th
56th	57th	58th	59th	60th	61st	62nd	63rd
64th	65th	66th	67th	68th	69th	70th	71st
72nd	73rd	74th	75th	76th	77th	78th	79th
80th	81st	82nd	83rd	84th	85th	86th	87th
88th	89th	90th	91st	92nd	93rd	94th	95th
96th	97th	98th	99th	100th	101st	102nd	103rd
104th	105th	106th	107th	108th	109th	110th	111th
112th	113th	114th	115th	116th	117th	118th	119th
120th	121st	122nd	123rd	124th	125th	126th	127th
128th	129th	130th	131st	132nd	133rd	134th	135th
136th	137th	138th	139th	140th	141st	142nd	143rd
144th	145th	146th	147th	148th	149th	150th	151st
152nd	153rd	154th	155th	156th	157th	158th	159th
160th	161st	162nd	163rd	164th	165th	166th	167th
168th	169th	170th	171st	172nd	173rd	174th	175th
176th	177th	178th	179th	180th	181st	182nd	183rd
184th	185th	186th	187th	188th	189th	190th	191st
192nd	193rd	194th	195th	196th	197th	198th	199th
200th	201st	202nd	203rd	204th	205th	206th	207th
208th	209th	210th	211st	212nd	213rd	214th	215th
216th	217th	218th	219th	220th	221st	222nd	223rd
224th	225th	226th	227th	228th	229th	230th	231st
232nd	233rd	234th	235th	236th	237th	238th	239th
240th	241st	242nd	243rd	244th	245th	246th	247th
248th	249th	250th	251st	252nd	253rd	254th	255th
256th	257th	258th	259th	260th	261st	262nd	263rd
264th	265th	266th	267th	268th	269th	270th	271st
272nd	273rd	274th	275th	276th	277th	278th	279th
280th	281st	282nd	283rd	284th	285th	286th	287th
288th	289th	290th	291st	292nd	293rd	294th	295th
296th	297th	298th	299th	300th	301st	302nd	303rd
304th	305th	306th	307th	308th	309th	310th	311st
312nd	313rd	314th	315th	316th	317th	318th	319th
320th	321st	322nd	323rd	324th	325th	326th	327th
328th	329th	330th	331st	332nd	333rd	334th	335th
336th	337th	338th	339th	340th	341st	342nd	343rd
344th	345th	346th	347th	348th	349th	350th	351st
352nd	353rd	354th	355th	356th	357th	358th	359th
360th	361st	362nd	363rd	364th	365th	366th	367th
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376th	377th	378th	379th	380th	381st	382nd	383rd
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464th	465th	466th	467th	468th	469th	470th	471st
472nd	473rd	474th	475th	476th	477th	478th	479th
480th	481st	482nd	483rd	484th	485th	486th	487th
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496th	497th	498th	499th	500th	501st	502nd	503rd
504th	505th	506th	507th	508th	509th	510th	511st
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544th	545th	546th	547th	548th	549th	550th	551st
552nd	553rd	554th	555th	556th	557th	558th	559th
560th	561st	562nd	563rd	564th	565th	566th	567th
568th	569th	570th	571st	572nd	573rd	574th	575th
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632nd	633rd	634th	635th	636th	637th	638th	639th
640th	641st	642nd	643rd	644th	645th	646th	647th
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656th	657th	658th	659th	660th	661st	662nd	663rd
664th	665th	666th	667th	668th	669th	670th	671st
672nd	673rd	674th	675th	676th	677th	678th	679th
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696th	697th	698th	699th	700th	701st	702nd	703rd
704th	705th	706th	707th	708th	709th	710th	711st
712nd	713rd	714th	715th	716th	717th	718th	719th
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776th	777th	778th	779th	780th	781st	782nd	783rd
784th	785th	786th	787th	788th	789th	790th	791st
792nd	793rd	794th	795th	796th	797th	798th	799th
800th	801st	802nd	803rd	804th	805th	806th	807th
808th	809th	810th	811st	812nd	813rd	814th	815th
816th	817th	818th	819th	820th	821st	822nd	823rd
824th	825th	826th	827th	828th	829th	830th	831st
832nd	833rd	834th	835th	836th	837th	838th	839th
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856th	857th	858th	859th	860th	861st	862nd	863rd
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880th	881st	882nd	883rd	884th	885th	886th	887th
888th	889th	890th	891st	892nd	893rd	894th	895th
896th	897th	898th	899th	900th	901st	902nd	903rd
904th	905th	906th	907th	908th	909th	910th	911st
912nd	913rd	914th	915th	916th	917th	918th	919th
920th	921st	922nd	923rd	924th	925th	926th	927th
928th	929th	930th	931st	932nd	933rd	934th	935th
936th	937th	938th	939th	940th	941st	942nd	943rd
944th	945th	946th	947th	948th	949th	950th	951st
952nd	953rd	954th	955th	956th	957th	958th	959th
960th	961st	962nd	963rd	964th	965th	966th	967th
968th	969th	970th	971st	972nd	973rd	974th	975th
976th	977th	978th	979th	980th	981st	982nd	983rd
984th	985th	986th	987th	988th	989th	990th	991st
992nd	993rd	994th	995th	996th	997th	998th	999th
1000th	1001st	1002nd	1003rd	1004th	1005th	1006th	1007th
1008th	1009th	1010th	1011st	1012nd	1013rd	1014th	1015th
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1024th	1025th	1026th	1027th	1028th	1029th	1030th	1031st
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1088th	1089th	1090th	1091st	1092nd	1093rd	1094th	1095th
1096th	1097th	1098th	1099th	1100th	1101st	1102nd	1103rd
1104th	1105th	1106th	1107th	1108th	1109th	1110th	1111st
1112nd	1113rd	1114th	1115th	1116th	1117th	1118th	1119th
1120th	1121st	1122nd	1123rd	1124th	1125th	1126th	1127th
1128th	1129th	1130th	1131st	1132nd	1133rd	1134th	1135th
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1152nd	1153rd	1154th	1155th	1156th	1157th	1158th	1159th
1160th	1161st	1162nd	1163rd	1164th	1165th	1166th	1167th
1168th	1169th	1170th	1171st	1172nd	1173rd	1174th	1175th
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1184th	1185th	1186th	1187th	1188th	1189th	1190th	1191st
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1208th	1209th	1210th	1211st	1212nd	1213rd	1214th	1215th
1216th	1217th	1218th	1219th	1220th	1221st	1222nd	1223rd
1224th	1225th	1226th	1227th	1228th	1229th	1230th	1231st
1232nd	1233rd	1234th	1235th	1236th	1237th	1238th	1239th

10. Integration

Indefinite integrals

eg) Evaluate $\int x^2 dx$ using the TI-89 by following these steps

[F3] 2 x^2, x, c [ENTER]

The \int tells the calculator to integrate with respect to x

The following should appear on your calculator screen.



Exercise.

Work out the answers to the following.

1. $\int x^3 dx$

2. $\int 2x^3 - 3x^2 + 5dx$

3. $\int (x+3)(x-17)dx$

4. $\int \frac{(2x^2 - 4x)}{2x} dx$

Find an antiderivative for each of the functions:

5. $-\frac{x^2}{\sqrt{2x}}$

6. $-e^{-4x}$

7. $\frac{\tan^2 x - 1}{\sin x}$

8. $\frac{-2e^{-4x} - 1}{3e^{2x}}$

Answers:

1. $\frac{x^4}{4} + c$

2. $\frac{x^4}{2} - x^3 + 5x + c$

3. $\frac{x^3}{3} - 7x^2 - 51x + c$

4. $\frac{x^2}{2} - 2x + c$

5. $\frac{-\sqrt{2}x^{\frac{5}{2}}}{5} + c$

6. $\frac{e^{-4x}}{4} + c$

7. $\frac{\cos x \ln(|\cos(x) + 1|) - \cos x \cdot \ln(|\cos(x) - 1|) + 2}{2 \cos(x)} + c$

8. $\frac{e^{-6x}(3e^{4x} + 2)}{18} + c$

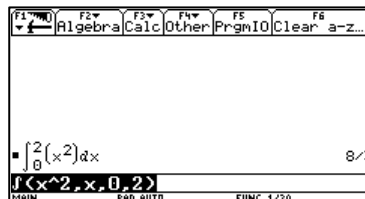
Definite Integrals

Evaluate the definite integral $\int_0^2 x^2 dx$ by following these steps

[F3] 2 x^2 , x , 0, 2) [ENTER]

0 = lower limit 2 = upper limit

The following should appear on your screen



Exercise. Work out these definite integrals

1. $\int_0^3 x^3 dx$

2. $\int_{-2}^1 2x^3 - 3x^2 + 5dx$

3. $\int_{-3}^{17} (x + 3)(x - 17)dx$

4. $\int_1^4 \frac{2x^2 - 4x}{2x} dx$

5. $\int_{\frac{\pi}{6}}^{\frac{3\pi}{2}} \sin(x) dx$

6. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos(x) dx$

7. $\int_{-\frac{1}{2}}^0 \left(\frac{e^{-2x} - 1}{e^{3x}} \right) dx$

Answers:

1. $\frac{81}{4}$

2. $\frac{-3}{2}$

3. $\frac{-4000}{3}$

4. $\frac{3}{2}$

5. $\frac{\sqrt{3}}{2}$

6. $\frac{-\sqrt{3}}{2} + 1$

7. $\frac{3e^{\frac{3}{2}} - 5e^{\frac{3}{2}} + 2}{15}$

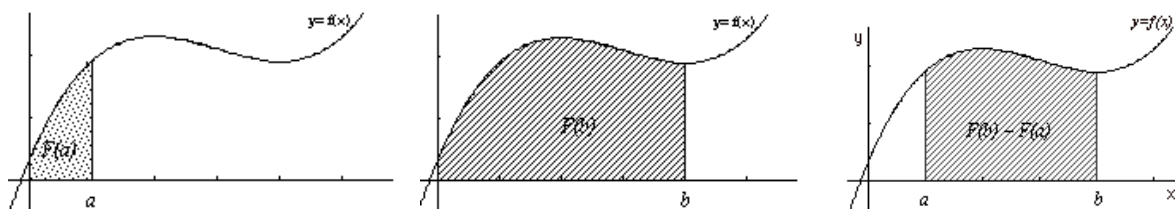
Definite Integrals as Areas

A definite integral written as $\int_a^b f(x) dx$ finds the area between the curve $f(x)$ and the x -axis, bounded by the lines $x = a$ and $x = b$.

$x = a$ is called the lower limit and $x = b$ is called the upper limit

An alternative method to calculating definite integrals is to graph the function first and then use the $\int f(x) dx$ facility.

We write $\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$ where $F(x)$ is the antiderivative of $f(x)$



Area from a to $b = F(b) - F(a)$

Total area is $\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$

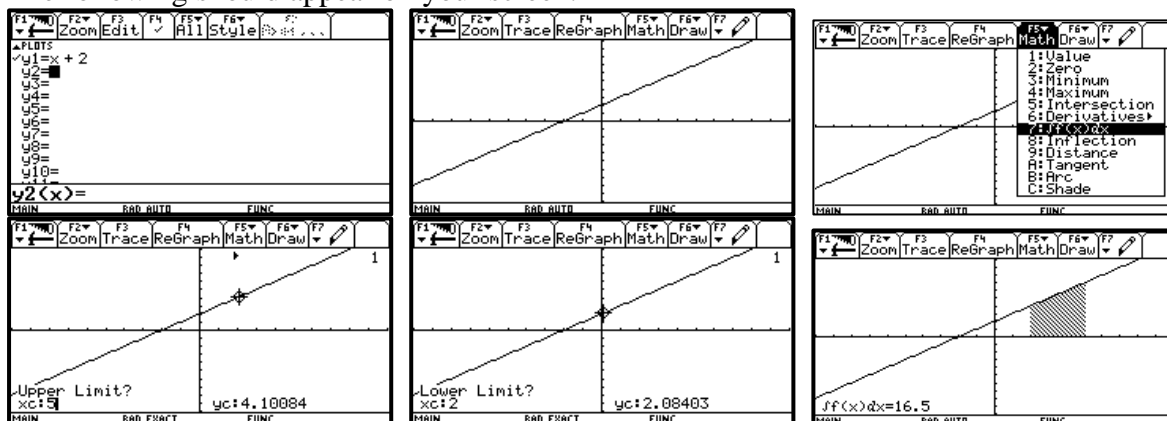
Follow these instructions to find this definite integral $\int_2^5 x + 2 dx$.

This method uses the graph of $f(x)$ to show the area represented by the integral and numeric integration to calculate it.

◆ [y=] x^2 [ENTER] [F2] 4 [F5] 7

Note : Only the x value of the lower and upper limit needs to be typed in. Ignore the y -value.

The following should appear on your screen.



Exercise

Follow the above method to represent these integrals as areas between the curve and the x -axis and calculate an answer for the definite integral. Use $y_1 =$ each time.

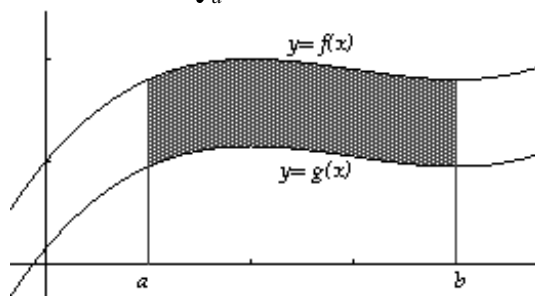
1. $\int_{-2}^2 x - 2 dx$
2. $\int_{-3}^0 x^2 + 3x dx$
3. $\int_2^3 (x + 3)(x - 2) dx$
4. $\int_{-2}^2 4 - x^2 dx$
5. $\int_1^3 4 - x^2 dx$

Answers:

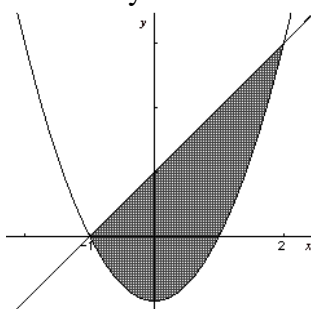
1. -8
2. -4.5
3. 2.83
4. 10.67
5. -0.666667

Area between two functions

$$\text{Area} = \int_a^b \{f(x) - g(x)\} dx$$

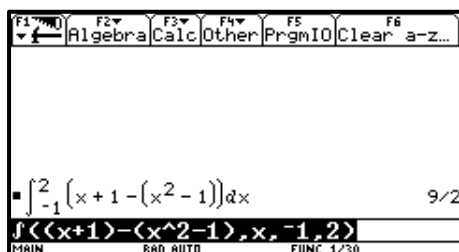
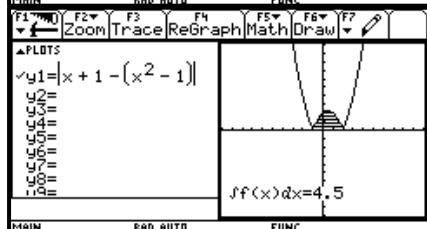
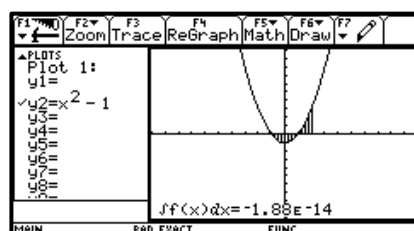
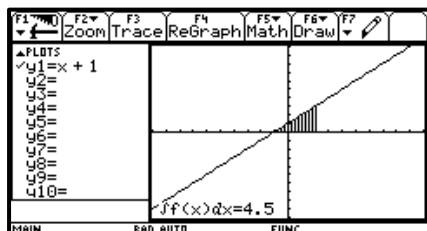


Example 1. Find area between $y = x + 1$ and $y = x^2 - 1$

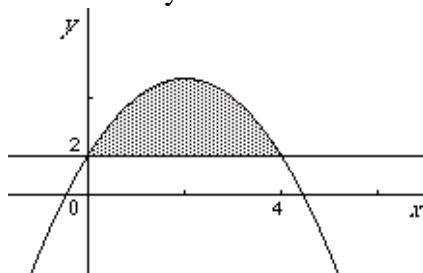


$$S = \int_{-1}^2 \{(x+1) - (x^2-1)\} dx = \int_{-1}^2 (-x^2 + x + 2) dx = \frac{9}{2} = 4.5$$

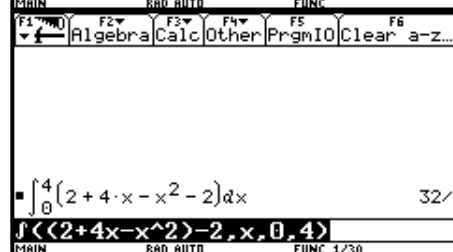
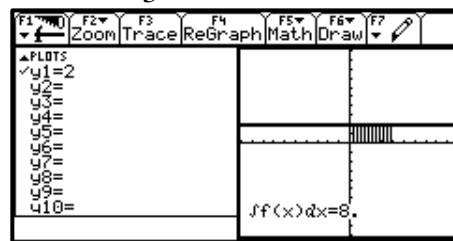
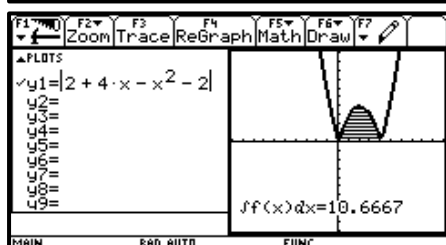
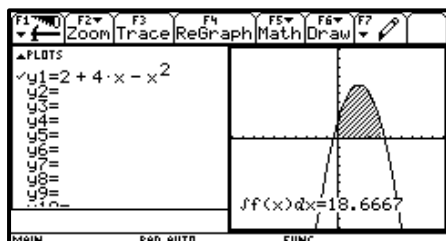
For functions with complex intersections we can use $\int_a^b |f(x) - g(x)| dx$



Example 2. Area between $y = 2 + 4x - x^2$ and $y = 2$

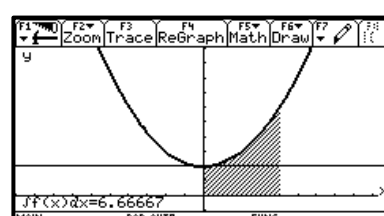
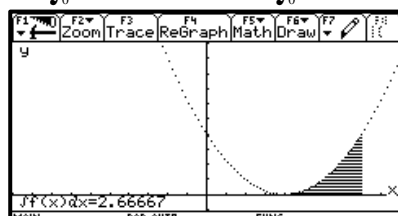
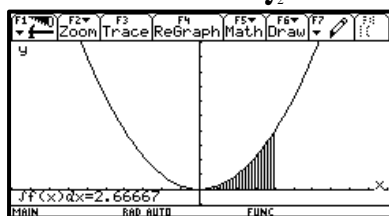


$$S = \int_0^4 \{(2 + 4x - x^2) - 2\} dx = \int_0^4 (4x - x^2) dx = \frac{32}{3}$$



Application

Example. Compare $\int_2^4 (x-2)^2 dx$ and $\int_0^2 (x^2 + 2) dx$ with $\int_0^2 x^2 dx$

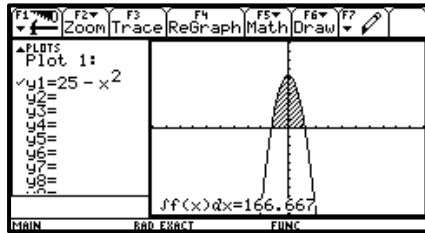


Exercises.

- Find the area enclosed by the curve $y = 25 - x^2$ and the x -axis. Sketch the graph and shade in this enclosed region. Write down the calculation you need to do to work out the area.
Hint : Use \blacksquare [F2] to adjust the window range for y-max
- Find the area enclosed by the curve $y = x^2 - 4x - 5$ and the x -axis. Sketch the graph and shade in this enclosed area. What does the negative sign indicate?
- Find the area enclosed by the parabola $y = (x - 2)^2$, the x -axis and the line $x=4$.
- Find the area bounded by the curve $y = x^2 - x + 2$ and the line $y = 8$.
- The function $f(x) = x(x + 1)(x - 2)$
 - Find the area bound by the curve, the x -axis and the lines $x = -1$ and $x = 0$.
 - Find the area bound by the curve, the x -axis and the lines $x = 0$ and $x = 2$.
 - Calculate $\int_{-1}^2 x(x + 1)(x - 2) dx$.
 - Explain why the answer to c) is not equal to the sum of the 2 areas found in a) and b).
- Using $f(x) = -x^3 + x^2 + 2x$ and $f(x-a)$ for $a = 0, 0.5, 1, 1.5, 2$ show that $\int_a^{1+a} f(x-a) dx$ is constant, and find its value.
 - Find a formula for $\int_0^k (f(x) + b) dx$ and demonstrate it graphically.

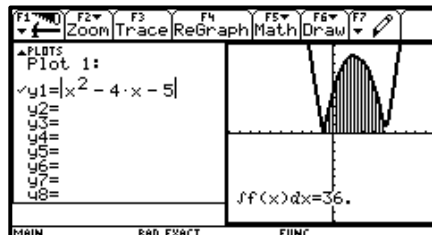
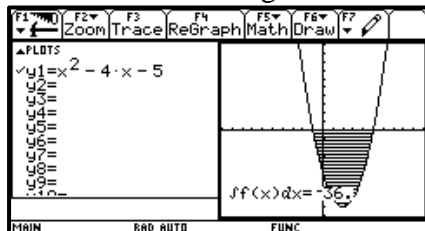
Answers:

1. $\int_{-5}^5 25 - x^2 dx = 166.667$

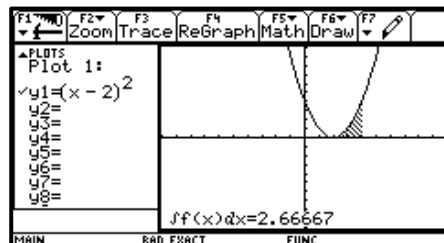


2. $\int_{-1}^5 (x^2 - 4x - 5) dx = -36$

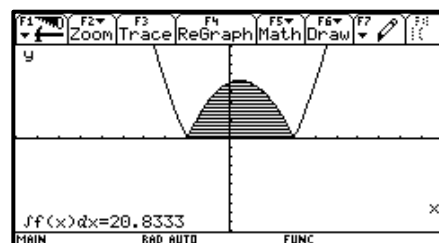
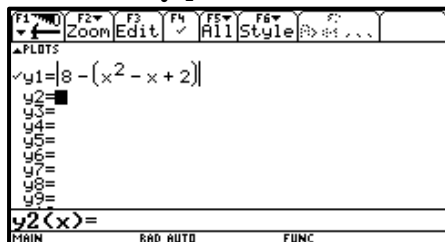
Area = 36 Negative indicates the area is below the x -axis



3. $\int_2^4 (x-2)^2 dx = 2.67$



4. Area = $\int_{-2}^3 \{8 - (x^2 - x + 2)\} dx = 20.8333$



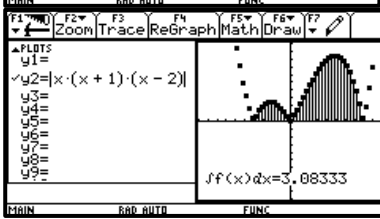
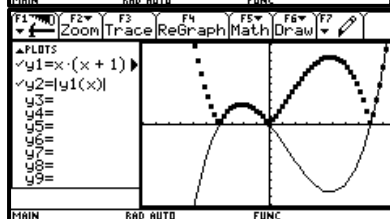
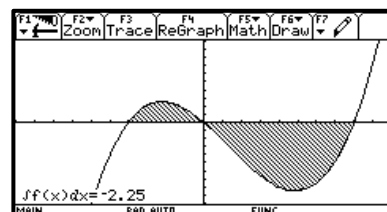
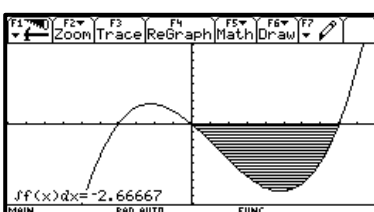
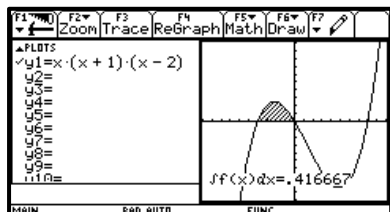
5. a) 0.4167

b) -2.667

c) -2.25

d) Integral does not always equal area. Integrals can be negative. Area is always positive.

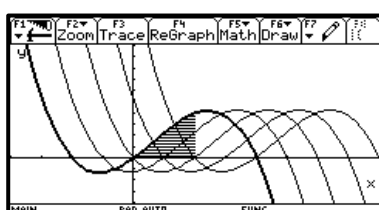
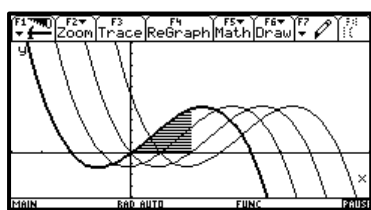
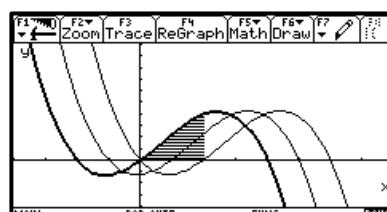
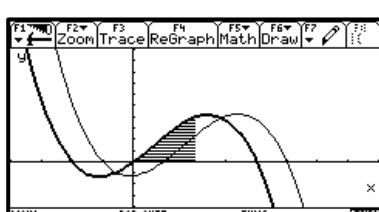
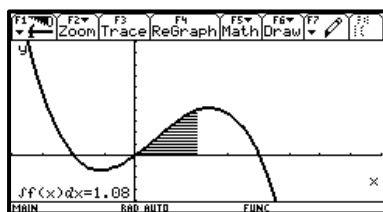
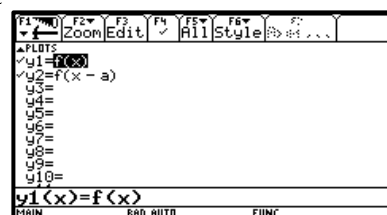
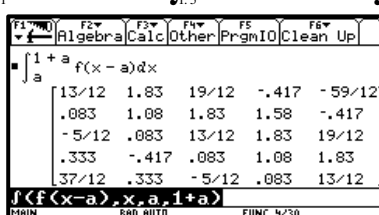
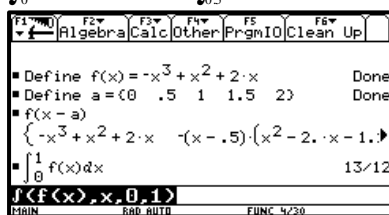
Area = $0.4167 + |-2.667| = 3.0837$



a) Define the function $f(x) = -x^3 + x^2 + 2x$ and $a = \{0, 0.5, 1, 1.5, 2\}$. We can see that $f(x-a)$ is transformed parallel to the x -axis. When we look at the $\int_a^{1+a} f(x-a) dx$ along the diagonal of the

results, then the values are all the same, that is;

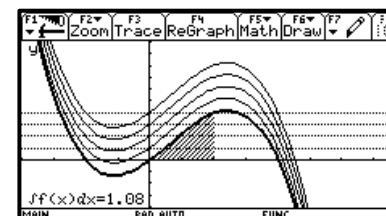
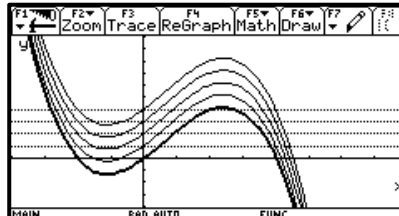
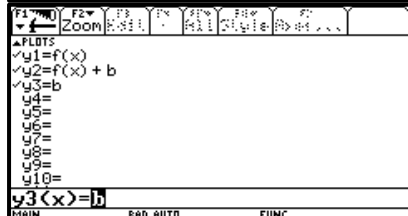
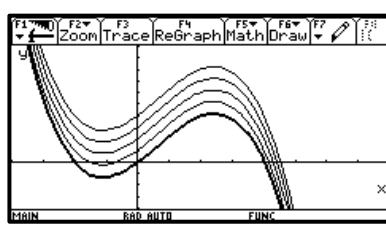
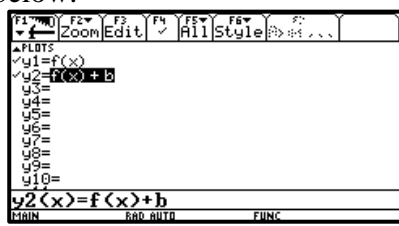
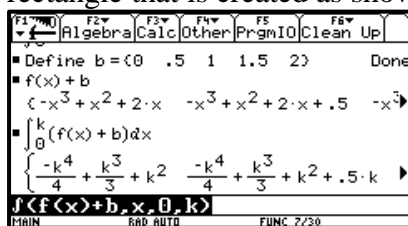
$$\int_0^1 f(x-0)dx = \int_{0.5}^{1.5} f(x-0.5)dx = \int_1^{2.5} f(x-1)dx = \int_{1.5}^{2.5} f(x-1.5)dx = \int_2^3 f(x-2)dx = 13/12 = 1.08.$$



b) If we define the value of $b = \{0, 0.5, 1, 1.5, 2\}$ then $f(x) + b$ is represented by:

$$-x^3 + x^2 + 2x, -x^3 + x^2 + 2x + 0.5, -x^3 + x^2 + 2x + 1, -x^3 + x^2 + 2x + 1.5, -x^3 + x^2 + 2x + 2.$$

From $\int_0^k (f(x) + b)dx$ we can see that the values of $0.5k, k, 1.5k, 2k$ represent area of the extra rectangle that is created as shown below.



11. Matrices

A matrix is simply a convenient way of storing data in an orderly number so that we know the exact position of any piece of data by reference to its row and column.

A matrix is a rectangular array of numbers of the form: A matrix with m rows and n columns is called $m \times n$

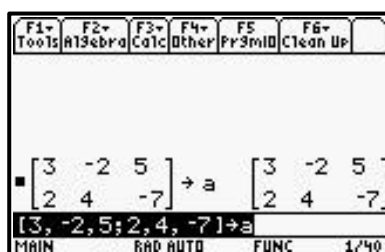
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The order of the matrix is determined by the number of rows and columns it contains.

Example. Determine a 2×3 matrix and represent it as A.

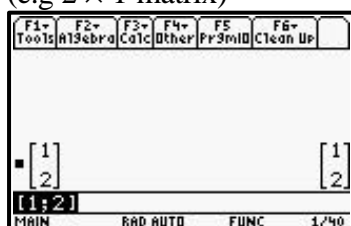
[3, -2, 5; 2, 4, -7] **STO►** **ALPHA** A

Note: The colon (;) separates rows.



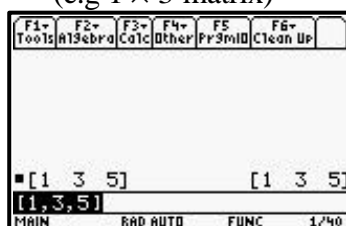
Column matrix

(e.g 2×1 matrix)



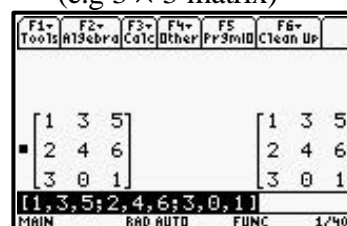
Row matrix

(e.g 1×3 matrix)



Square matrix

(e.g 3×3 matrix)

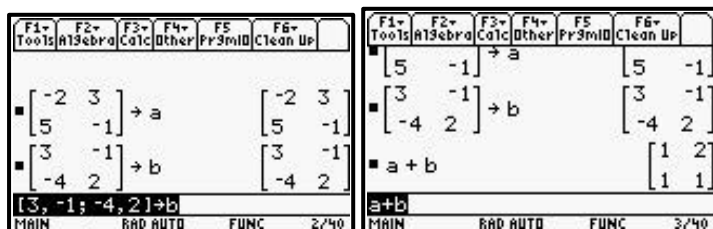


Addition of Matrices

Matrices are added by adding elements in corresponding positions.

Matrices can only be added if they are of the same order.

Example. If $A = \begin{bmatrix} -2 & 3 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$ find $A + B$



$$\begin{bmatrix} -2 & 3 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -2+3 & 3+(-1) \\ 5+(-4) & (-1)+2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Multiplication of Matrices

1. Multiplication by a Scalar

To multiply a matrix by a (scalar) value we multiply every value in the matrix by that value.

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Example. Find $-2 \begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$.

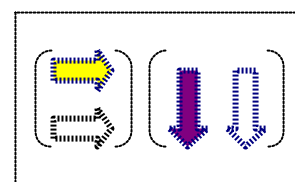
$$-2 \begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -6 \\ -8 & -2 & 0 \end{bmatrix}$$

2. Multiplication of a Matrix by a Matrix

- identify the position of the element in the product matrix; e.g. first row, second column
- multiply the elements in the appropriate row in the first matrix by the corresponding elements in the same column of the second matrix.

The product of two 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$



Example. If $A = \begin{bmatrix} 2 & -1 & 4 \\ -1 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & -2 \\ 3 & 2 \end{bmatrix}$ find AB .

Order of $A = 2 \times 3$ and order of $B = 3 \times 2$ so order of $AB = 2 \times (3 \times 2) \rightarrow 2 \times 2$

- identify the position of the element in the product matrix; e.g. first row, second column
- multiply the elements in the appropriate row in the first matrix by the corresponding elements in the same column of the second matrix.

$$AB = \begin{bmatrix} 2 & -1 & 4 \\ -1 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 0 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot (-2) + (-1) \cdot 0 + 4 \cdot 3 & 2 \cdot 1 + (-1) \cdot (-2) + 4 \cdot 2 \\ (-1) \cdot (-2) + 1 \cdot 0 + 3 \cdot 3 & (-1) \cdot 1 + 1 \cdot (-2) + 3 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 12 \\ 11 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 1 \\ 0 & -2 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 4 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} (-2) \cdot 2 + 1 \cdot (-1) & (-2) \cdot (-1) + 1 \cdot 1 & (-2) \cdot 4 + 1 \cdot 3 \\ 0 \cdot 2 + (-2) \cdot (-1) & 0 \cdot (-1) + (-2) \cdot 1 & 0 \cdot 4 + (-2) \cdot 3 \\ 3 \cdot 2 + 2 \cdot (-1) & 3 \cdot (-1) + 2 \cdot 1 & 3 \cdot 4 + 2 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 & -5 \\ 2 & -2 & -6 \\ 4 & -1 & 18 \end{bmatrix}$$

From this we see that $AB \neq BA$ and so matrix multiplication is not commutative.

F1	F2	F3	F4	F5	F6
Tools	Algebra	Calc	Other	Pr3mID	Clean Up
[2 -1 4] → a [2 -1 4]					
[-1 1 3] → b [-1 1 3]					
[-2 1] → b [-2 1]					
[0 -2] → b [0 -2]					
[3 2] → b [3 2]					
[-2, 1; 0, -2; 3, 2] → b					
MAIN	RAD	AUTO	FUNC	7/40	

F1	F2	F3	F4	F5	F6
Tools	Algebra	Calc	Other	Pr3mID	Clean Up
[-2 1] → b [-2 1]					
[0 -2] → b [0 -2]					
[3 2] → b [3 2]					
a · b [8 12]					
a * b [11 3]					
MAIN	RAD	AUTO	FUNC	8/40	

F1	F2	F3	F4	F5	F6
Tools	Algebra	Calc	Other	Pr3mID	Clean Up
a · b [8 12]					
b · a [-5 3 -5]					
b * a [2 -2 -6]					
[4 -1 18]					
MAIN	RAD	AUTO	FUNC	9/40	

We need the multiplication sign (*) in AB and BA : $A*B$ and $B*A$

Identity matrix

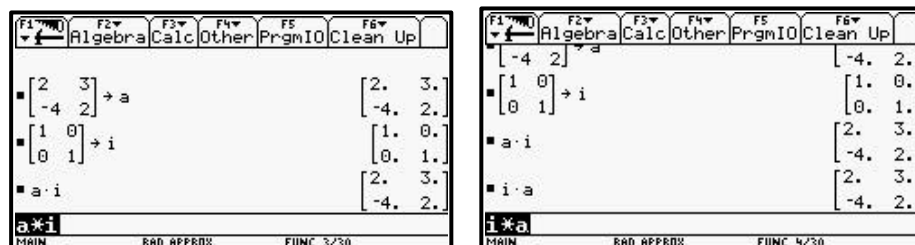
This is defined as that matrix I for which

$$AI = IA = A$$

This can only happen for $n \times n$ square matrices, with the same number of rows and columns (why?).

Consider

$$\begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$$



In general the identity matrix is that $n \times n$ matrix with 1s down the diagonal and zeros elsewhere.

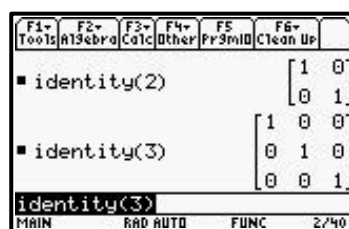
On the TI-89 this is obtained by :

$\boxed{2\text{nd}} \boxed{5}$ Option 4: Matrix ? 6: identity (2) $\boxed{\text{ENTER}}$

then type n) for an $n \times n$ identity. NB n must be a value!



For example:



Transpose

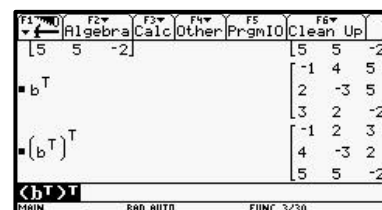
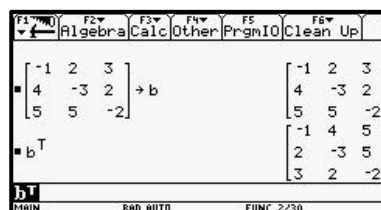
The transpose A^T of a matrix A is a matrix formed by interchanging the rows and columns of A .

$\begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix} \boxed{\text{STO}} \boxed{a} \boxed{\text{ENTER}}$ a $\boxed{2\text{nd}} \boxed{5}$ Option: 4 Option: 1 (T) $\boxed{\text{ENTER}}$

$$\begin{pmatrix} 2 & 3 \\ -4 & 2 \end{pmatrix}^T = \begin{pmatrix} 2 & -4 \\ 3 & 2 \end{pmatrix}$$



$$\begin{pmatrix} -1 & 2 & 3 \\ 4 & -3 & 2 \\ 5 & 5 & -2 \end{pmatrix}^T = \begin{pmatrix} -1 & 4 & 5 \\ 2 & -3 & 5 \\ 3 & 2 & -2 \end{pmatrix}$$



Clearly $(A^T)^T = A$

And also

$$(A + B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T \quad \text{note order!}$$

Example

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \text{ then } A^T = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} \text{ then } B^T = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}^T \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 13 \\ 8 & 6 \end{pmatrix}$$

$$(AB)^T = \left(\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} \right)^T = \begin{pmatrix} -1 & 8 \\ 13 & 6 \end{pmatrix}^T = \begin{pmatrix} -1 & 13 \\ 8 & 6 \end{pmatrix}$$



Determinant of Matrix

For square matrices, of size $n \times n$ we can define a determinant, which will help us find its inverse. The determinant of a 2×2 matrix A:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is written

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

and is given by $\det A = ad - bc$.

On the TI-89 this is obtained by :

2nd 5 Option:4 (Matrix)

Option:2 det ([a, b ; c , d]) **ENTER**



Determinant can be used at the start of a problem on simultaneous equations to check for consistency.

Example. Solve the following sets of simultaneous equations.

a) $x + y = 4$

b) $2x - y = 3$

c) $4x - 3y = 12$

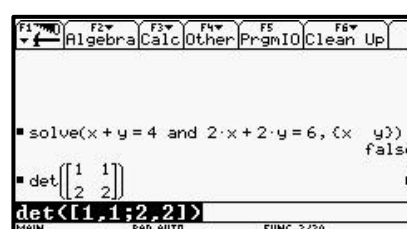
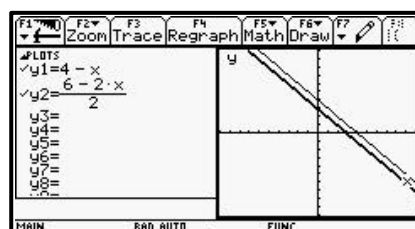
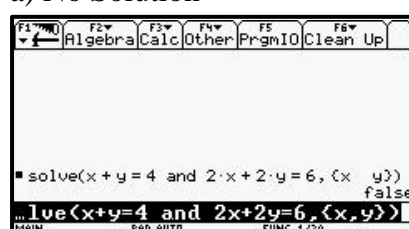
$2x + 2y = 6$

$4x - 2y = 6$

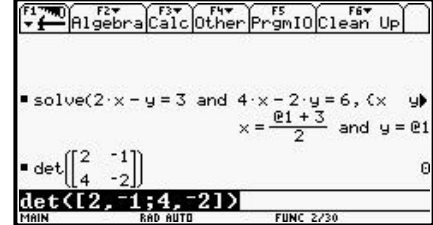
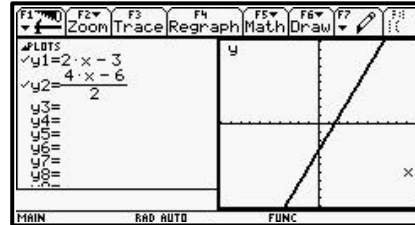
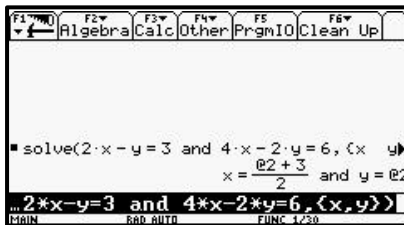
$x - 2y = -2$

Sol)

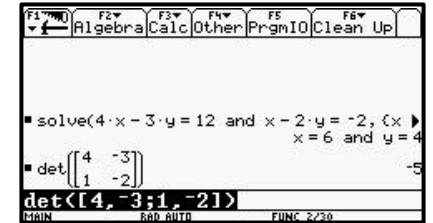
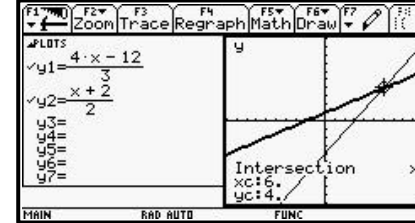
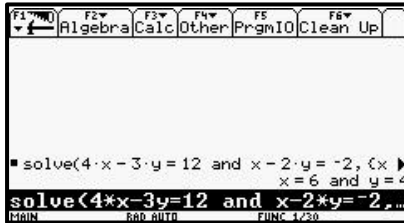
a) No Solution



b) Infinitely many solutions

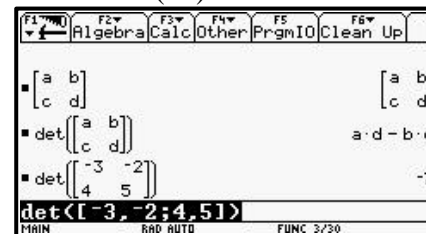


c) Unique solution: $x = 6$ and $y = 4$



For example

$$\begin{vmatrix} -3 & -2 \\ 4 & 5 \end{vmatrix} = -15 - (-8) = -15 + 8 = -7$$



The determinant of a 3×3 matrix A:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

? Multiply the 3 numbers on each of the leading diagonals (from top left to bottom right): add together from this total, subtract the sum of the products on the other 3 diagonals.

? $\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$, $\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$ and $\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$ are called **minors**. The minor of an element in a determinant is the determinant formed by omitting the row and column in which the element occurs.

The cofactor of an element is its minor together with its sign. The signs for 3×3 matrix are

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

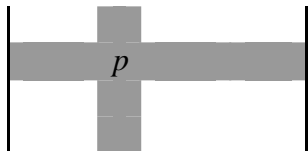
Co-factors

The co-factor of an element in a determinant is the determinant of that the matrix obtained by removing the row and column containing the element from the original determinant, multiplied by +1 or -1 according to the position of the element, as below

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

In general the sign is given by $(-1)^{i+j}$ where the value is in the i th row and the j th column.

For example, remove the shaded cells below to get the cofactor values



Then the value of the determinant for an $n \times n$ matrix is given by

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \dots + a_{1n}C_{1n}$$

For example, given matrix $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12, \quad c_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = 16$$

Adjoint

The matrix of the cofactors of the transpose of a matrix A: $\text{adj } A$

Example. Evaluate the determinant, the cofactor and adjoint of matrix A

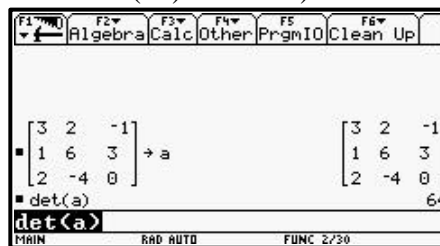
$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\det A = (3 \times 6 \times 0 + 2 \times 3 \times 2 + (-1) \times 1 \times (-4)) - ((-1) \times 6 \times 2 + 3 \times 3 \times (-4) + 2 \times 1 \times 0) = 64$$

On the TI-89 this is obtained by :

[3, 2, -1 ; 1, 6, 3; 2, -4, 0] [STO] a [ENTER]

[2nd] 5 Option:4 (Matrix) Option:2 det (a) [ENTER]



$$\text{Cofactor of matrix A} = \begin{bmatrix} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -2 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

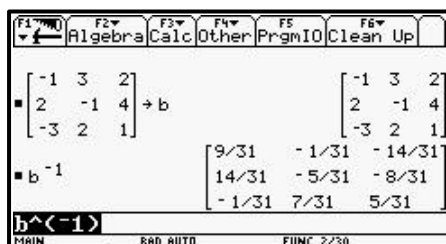
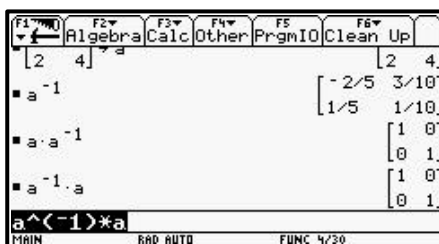
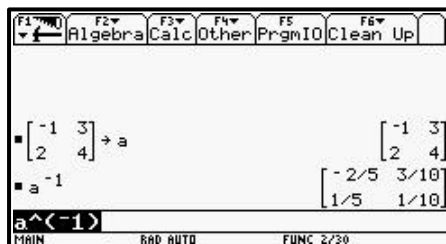
Inverses

We define the inverse of a matrix A to be that matrix A^{-1} such that:

$$A A^{-1} = A^{-1} A = I_n$$

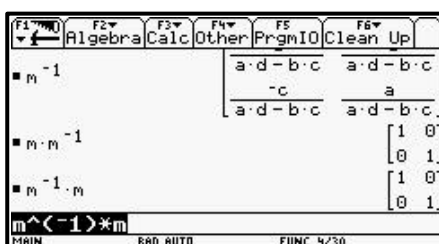
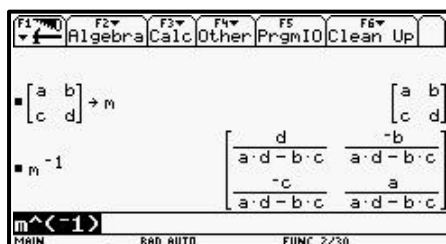
Where I_n is the $n \times n$ identity matrix.

Thus only square matrices can have inverses.



Inverses for 2x2 matrices

For 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$



To solve the simultaneous equations as a single matrix equation:

Write the system of equation as a single matrix equation

$$\begin{matrix} ax + by = c \\ cx + dy = f \end{matrix} \quad \text{becomes} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

If we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

And A^{-1} be the inverse of A , then we can multiply both sides of equation 1 by A^{-1} , giving

$$A^{-1} A \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$

But $A A^{-1} = I$ by definition

So
$$I \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$

And hence the solution is
$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} e \\ f \end{bmatrix},$$

where
$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

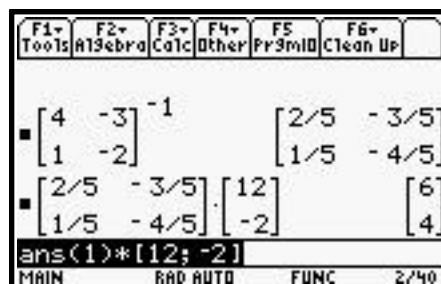
To solve simultaneous equations we multiply the number matrix by the inverse of the coefficient matrix, if it exists (if $\det A \neq 0$).

If the matrix of coefficient is singular (the determinant $ad - bc = 0$), the simultaneous equations represent either two parallel lines or two lines which are coincident.

Example: Solve the simultaneous equations $4x - 3y = 12$ and $x - 2y = -2$.

In matrix form: $\begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ -2 \end{bmatrix} \\ &= \frac{1}{4 \cdot (-2) - (-3) \cdot 1} \begin{bmatrix} -2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 12 \\ -2 \end{bmatrix} \\ &= -\frac{1}{5} \begin{bmatrix} -2 \cdot 12 + 3 \cdot (-2) \\ (-1) \cdot 12 + 4 \cdot (-2) \end{bmatrix} \\ &= -\frac{1}{5} \begin{bmatrix} -30 \\ -20 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 4 \end{bmatrix} \end{aligned}$$



So $x = 6$, $y = 4$

Inverses for 3x3 matrices

Use matrix methods to solve

$$\begin{aligned} 3x - y + 2z &= 13 \\ -x + 4y + 2z &= -1 \\ 4y + 3z &= 4 \end{aligned}$$

Writing in matrix form: $\begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 2 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \\ 4 \end{bmatrix}$

To find the inverse of a 3x3 matrix we carry out the following steps.

Given a matrix A, for example, $A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 2 \\ 0 & 4 & 3 \end{bmatrix}$

Step 1. Define A^T , the transpose of A

$$A^T = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & 4 \\ 2 & 2 & 3 \end{bmatrix}$$

Step 2. Obtain the *adjoint* matrix, written $\text{adj } A$, by replacing each element in the transpose of A by its cofactor, and by changing the sign of every second element.

$$\text{adj } A = \begin{bmatrix} \begin{vmatrix} 4 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 4 \end{vmatrix} & \begin{vmatrix} -1 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 11 & -10 \\ 3 & 9 & -8 \\ -4 & -12 & 11 \end{bmatrix}$$

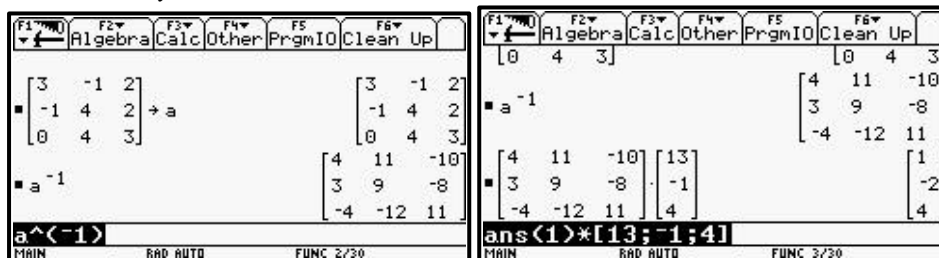
Step 3. Find $\det A$

$$\det A = (3 \times 4 \times 3 + (-1) \times 2 \times 0 + 2 \times (-1) \times 4) - (2 \times 4 \times 0 + 2 \times 4 \times 3 + 3 \times (-1) \times (-1)) = 36 - 8 - (24 + 3) = 1$$

$$A^{-1} = \frac{1}{\det A} \cdot \text{adj}(A) = \frac{1}{1} \begin{bmatrix} 4 & 11 & -10 \\ 3 & 9 & -8 \\ -4 & -12 & 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 11 & -10 \\ 3 & 9 & -8 \\ -4 & -12 & 11 \end{bmatrix} \begin{bmatrix} 13 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Thus $x = 1$, $y = -2$ and $z = 4$



Systems of linear equations

A solution to a system of linear equations gives the corresponding values of each of the variables that satisfy all the equations simultaneously.

Solving Systems of Equations

There are three things we can do to a system of equations which do not alter their solutions:

1. Interchange any two equations
2. Multiply any equation through by a constant ($\neq 0$)
3. Add a constant multiple ($\neq 0$) of any equation to any other equation.

Gaussian Elimination

When there are 3 equations – in x , y , and z – we start by eliminating the first variable (x) in the last 2 equations and then eliminate the second variable (y) in the last equation. This leaves us with a set of equations in *upper triangular form*, or **echelon form**.

Once the equations are *echelon form*, they can be solve by **back substitution**.

The leading variable in each equation in the list falls further to the right each time.

The rules become:

1. Interchange any two rows
2. Multiply any row through by a constant ($\neq 0$)
3. Add a constant multiple ($\neq 0$) of any row to any other row.

Example.

$$\begin{aligned} 3x - y + 2z &= 13 \dots\dots\dots R1 \\ -x + 4y + 2z &= -1 \dots\dots\dots R2 \\ 4y + 3z &= 4 \dots\dots\dots R3 \end{aligned}$$

The augmented matrix is: $\begin{bmatrix} 3 & -1 & 2 & 13 \\ -1 & 4 & 2 & -1 \\ 0 & 4 & 3 & 4 \end{bmatrix}$

To make echelonform, $[3, -2, 2, 13 ; -1, 4, 2, -1 ; 0, 4, 3, 4]$ **[STO]** a **[ENTER]**
 $R1 + 3R2$

$$\begin{bmatrix} 3 & -1 & 2 & 13 \\ 0 & 11 & 8 & 10 \\ 0 & 4 & 3 & 4 \end{bmatrix}$$

$$-4R_2 + 11R_3$$

$$\begin{bmatrix} 3 & -1 & 2 & 13 \\ 0 & 11 & 8 & 10 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} 3x - y + 2z &= 13 \\ 11y + 8z &= -1 \\ z &= 4 \end{aligned}$$

By back substitution,

$$z = 4,$$

$$11y + 8(4) = -1, \text{ so } y = -2$$

$$3x - (-2) + 2(4) = 13, \text{ so } x = 1$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & \frac{3}{4} & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

2nd 5 Option: 4 (Matrix) J (Row ops) 3(mRow)

(3, a, 2) **ENTER**

2nd 5 Option: 4 J (Row ops) 4 (mRowAdd (1, ans(1), 1, 2) **ENTER**

On the TI-89 this is obtained by:

[3, -2, 2, 13; -1, 4, 2, -1; 0, 4, 3, 4] **STO>** a **ENTER**

2nd 5 Option 4 (Matrix) 3(ref) (a) **ENTER**