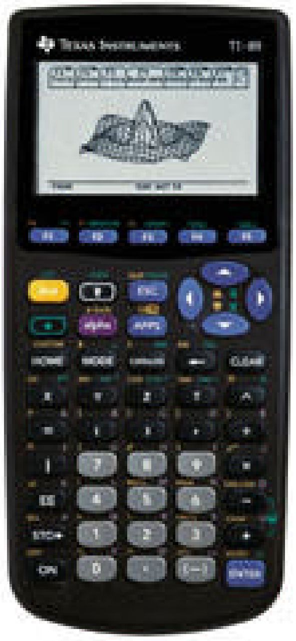
THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

TI-89 Workshop

Algebra and Calculus



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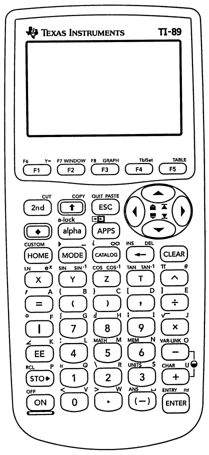
The University of Auckland

24th – 28th February 2003

1. Saying 'Hello' to your CAS calculator



You will use the following keys.

- Press [ON]

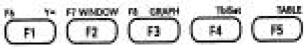
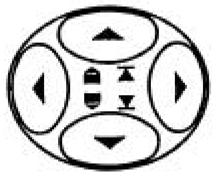
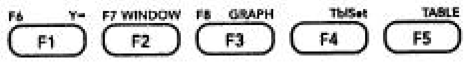
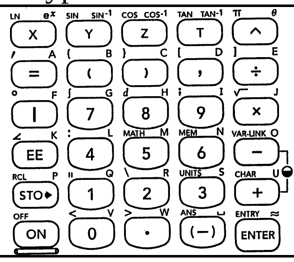
The calculator cursor should be in the Home Screen (see the black cursor flashing in the bottom left hand corner).

- Press [2nd] [ON]

The calculator should turn off.

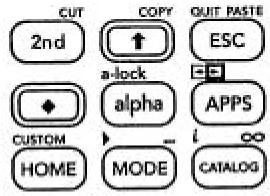
- If you can't see the screen use [◆] [−] (lighter) or [◆] [+] (darker) to change screen contrast.
- [HOME] displays the Home Screen, where you perform most calculations.

Basic Facilities of the TI-89

Function Keys	Cursor Pad
[F1] through [F8] function keys let you select toolbar menus. 	The cursor is controlled by the large blue circle on the top right hand side of the calculator. This allows access to any part of an expression. 
Application Short Keys	**Calculator Keypad**
Used with the [◆] key to let you select commonly used applications: [Y=] [WINDOW] [GRAPH] [TblSet] [TABLE] 	Performs a variety of mathematical and scientific operations 

[2nd] [◆] [⬆] and [ALPHA] modify the action of other keys:

Modifier	**Description**
[2nd] (Second)	Accesses the second function of the next key you press
[◆] (Diamond)	Activates "shortcut" keys that select applications and certain menu items directly from the keyboard.
[⬆] (Shift)	Types an uppercase character for the next letter key you press.
[ALPHA]	Used to type alphabetic letters, including a space character. On the keyboard, these are printed in the same colour as the [ALPHA] key.

	Key	Description
	[APPS]	Displays a menu that lists all the applications available on the TI-89.
	[ESC]	Cancels any menu or dialogue box.
	[ENTER]	Evaluates an expression, executes an instruction, selects a menu item etc…
	[MODE]	Displays a list of the TI-89's current mode settings, which determine how numbers and graphs are interpreted, calculated, and displayed.
	[CLEAR]	Clears (erases) the entry line.
	[CATALOG]	Press ↑ or ↓ to move the cursor to the function or instruction. (You can move quickly down the list by typing the first letter of the item you need.) Press [ENTER] Your selection is pasted on the home screen.

Application	Lets you:
[Home]	Enter expressions and instructions, and performs calculations
[Y=]	Define, edit, and select functions or equations for graphing
[Window]	Set window dimensions for viewing a graph
[Graph]	Display graph
[Table]	Display a table of variable values that correspond to an entered function

Press:	To display
[F1] [F2]… etc.	A toolbar menu– Drops down from the toolbar at the top of most application screens. Lets you select operations useful for that application
[2nd] [CHAR]	CHAR menu– Lets you select from categories of special characters (Greek, math, etc.)
[2nd] [MATH]	MATH menu– Lets you select from categories of mathematical operations

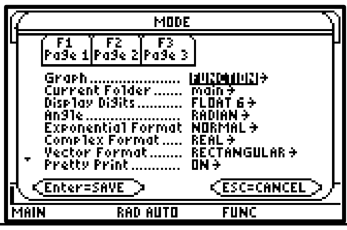
- [2nd] [F6] Clean Up to start a new problem:

Clear a–z	Clears (deletes) all single-character variable names in the current folder. If any of the variables have already been assigned a value, your calculation many produce misleading results.

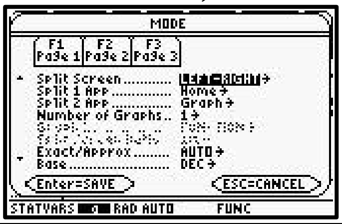
Problem?	*Try this!*
If you make a typing error	If you make a typing error use [←] to undo one character at a time If necessary, press [CLEAR] to delete the complete line.
If you want to clear the home screen completely	Press F1 [8]

Mode Settings

Press [MODE], this shows the modes and their current settings



If you press **F2** then 'Split Screen' specifies how the parts are arranged: FULL (no split screen), TOP-BOTTOM, or LEFT-RIGHT



(a) Entering a Negative Number

Use [−] for subtraction and use [(-)] for negation.

To enter a negative number, press [(-)] followed by the number.

To enter the number –7, press [(-)] 7.

9 × [(-)] 7 = –63,

9 × [−] 7 = displays an error message

To calculate –3 – 4, press [(-)] 3 [−] 4 [ENTER]

(b) Implied Multiplication

If you enter:	The TI-89 interprets it as:
2*a*	2**a*
xy	Single variable named *xy*; CAS does not read as $x \times y$

(c) Substitution

Using [|] key to find the value of a function or expression given particular values of a variable

eg) *x*^2+2 [|] *x*=3

This calculates the value of $x^2 + 2$ given $x = 3$

Using 'STORE' key: [STO▸]

eg) Find *f*(2) if $f(x) = -x^3 + 2$

-*x*^3+2 [STO▸] *f*(*x*)	$-x^3 + 2 \rightarrow f(x)$
f(2)	–6

(d) Rational Function Entry

$\frac{f(x)}{g(x)} = \frac{(f(x))}{(g(x))}$ **= (numerator) [÷] (denominator)**

For example, $\frac{x+1}{2x-1} \rightarrow (x + 1)$ [÷] $(2x - 1)$

(e) Operators

addition: + subtraction : – multiplication: × division: ÷ Exponent: ^

(f) Elementary Functions

exponential: e^(*x*) natural logarithm: ln(*x*) square root: √ absolute value: abs(*x*)

trigonometric: sin(*x*), cos(*x*), tan(*x*), $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$

If you want <u>sec(*x*)</u> then put 1/cos(*x*) or use the catalogue: [CATALOG] [3] [ENTER], cosec(*x*) is 1/sin(*x*).

Note: The trigonometric functions in TI-89 angles are available in both degrees and radians. If you want degrees (180°) or radians (π) change using the 3 key previously discussed.

(g) Constants

i : imaginary number	π : Pi	∞ : infinity
with [2nd] [CATALOG] **key**	**with** [2nd] [^] **key**	**with** [◆] [CATALOG] **key**

(h) Recalling the last answer

2nd [ANS]

ex) ans(1) Contains the last answer
ans(2) Contains the next-to-last answer

(i) Cutting, Copying and Pasting

Use ◀ ▶ or ▼ ▲ to highlight an expression.
Press [F1]5, to copy and [F1]6 to paste.
Press ENTER to replace the contents of the entry line with any previous entry.

(j) When differentiating with respect to *x*

Limit $\lim_{x \to a} f(x)$: **lim(*f(x)*, *x* , *a*)**

Differentiation $\frac{d}{dx} f(x)$: *d*(*f*(*x*) , *x*)

Indefinite Integral $\int f(x)dx$: ∫ **(*f(x)* , *x*, *c*)**

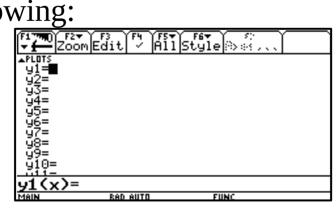
Definite integral $\int_a^b f(x)\,dx$: ∫ **(*f(x)*, *x*, *a*, *b*)**

Area between *f(x)* and *g(x)* on the interval [*a*, *b*]: $\int_a^b |f(x) - g(x)|\, dx$

2. [Y=] and [Table]

(a) The [Y=] menu

Press ◆ [Y=] to see the following:

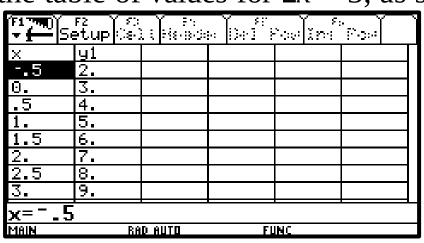


If there are any functions to the right of any of these eight equal signs, place the cursor on them (using the arrow keys) and press CLEAR
Place the cursor just to the right of y1= and follow the sequence below.

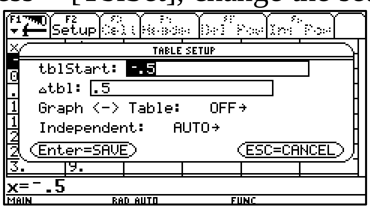
Press	See	Explanation
2*x* + 3	y1(*x*) = 2*x* + 3	You have entered y1 = 2*x* + 3
HOME		This returns you to a blank Home Screen.
y1(*x*) ENTER	y1(*x*) 2*x* + 3	This pastes y1 on the Home Screen.
y1(4) ENTER	y1(4) 11	This finds the value of y1 when *x* = 4.

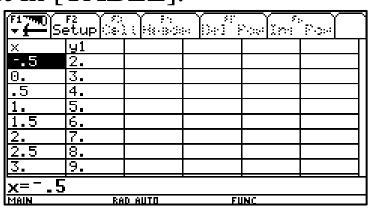
(b) Table

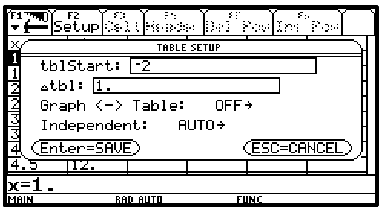
Press ◆ [TABLE] to see the table of values for 2*x* + 3, as shown below:

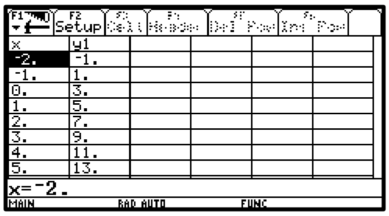


Press • [TblSet], change the settings and see the effect in [TABLE].





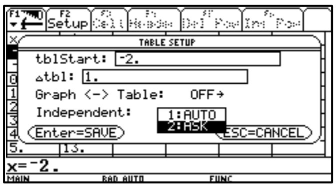


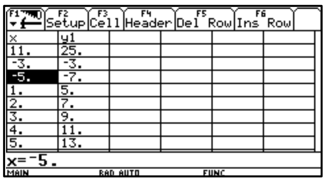


By changing [TblSet] from [1. AUTO] to [2. ASK], complete the table below:

x	y1
11	?
–3	?
–5	?

Remember: y1 is still set to 2*x* + 3

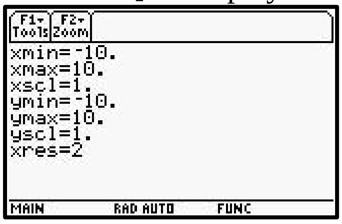


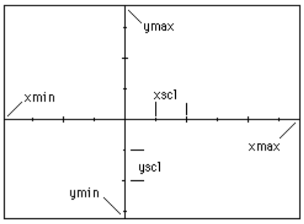


3. Graphing

(a) Displaying Window Variable in the Window Editor

Press ◆ **[WINDOW]** to display the Window Editor.

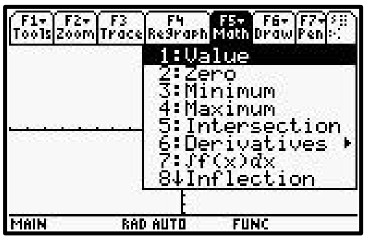




Variables	Description
*x*min, *x*max, *y*min, *y*max *x*scl, *y*scl *X*res	Boundaries of the viewing window. Distance between tick marks on the *x* and *y* axes. Sets pixel resolution (1 through 10) for function graphs. The default is 2.

(b) **Overview of the Math Menu**

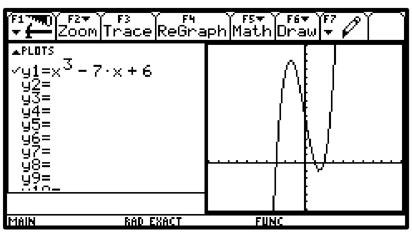
Press **F5** from the Graph screen

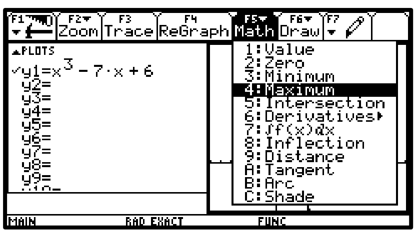


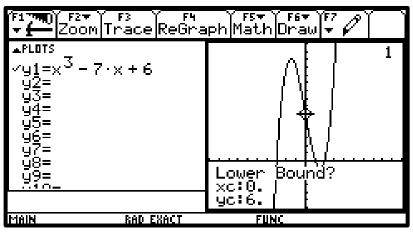
Math Tool	Description
Value Zero, Minimum, Maximum Intersection Derivatives $\int f(x)dx$ A:Tangent	Evaluates a selected *y*(*x*) function at a specified *x* value Finds a zero (*x*-intercept), minimum, or maximum point within an interval. Finds the intersection of two functions. Finds the derivative (slope) at a point. Finds the approximate numerical integral over an interval. Draws a tangent line at a point and displays its equation

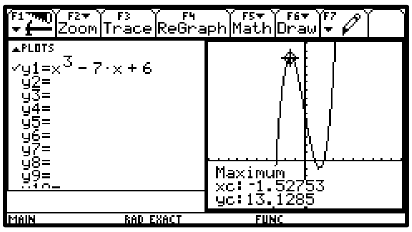
(c) **Finding the Maximum & Minimum Values of a Function from its Graph**

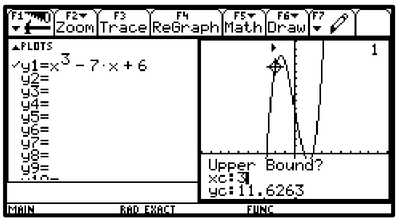
1. Display the **Y=Editor**.
2. Enter the function
3. Open the Math Menu **F5**, and select **4: Maximum**.
4. Set the lower bound.
5. Set the upper bound.
6. Find the maximum point on the graph between the lower and upper bounds.
7. Transfer the result to the Home screen, and then display the Home screen. HOME





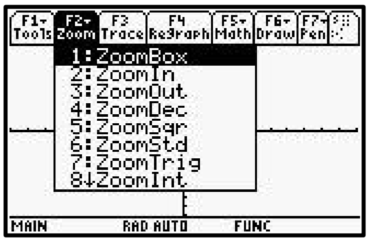






(d) **Overview of the Zoom Menu**

Press **F2** from ***y*=Editor**, window Editor, or Graph screen



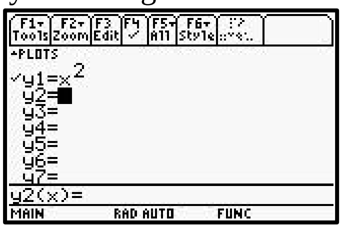
Zoom tool	**Description**
1:ZoomBox	Lets you draw a box and zoom in on that box.
2:ZoomIn 3:ZoomOut	Lets you select a point and zoom in or out by an amount defined by **SetFactors**.
4:ZoomDec	Sets Δx and Δy to 0.1, and centres the origin.
6:ZoomStd	Sets Window variables to their default values. *x*min= –10, *x*max= 10, *x*scl=1, *y*min= –10, *y*max= 10, *y*scl= 1, *x*res= 2

Notes:

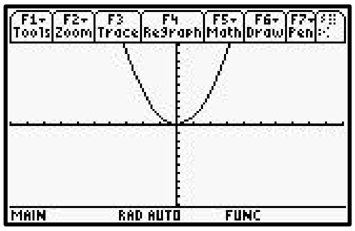
To get out of the graphing mode use [HOME]**.**
This will not work while the **BUSY** icon is flashing in the bottom right hand corner.
Adjust your graph by selecting **F2** and choosing **2:ZoomIn**, **3:ZoomOut**, or **A:ZoomFit**

eg) Graph $y = x^2$ by following these instructions.

◆ [y=] x ^ 2 [ENTER]

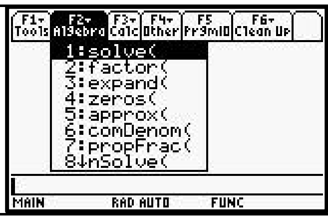
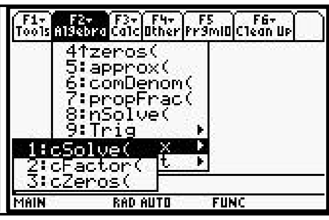


◆ [F3]



To draw a new graph go to [Y=] and change the formula in the ***y*1** position using the cursor to move up to it to delete it. This effectively clears the previous graph as well. Alternatively, using y2 will add the new graph to $y = x^2$.
[HOME] returns you to the Home screen.

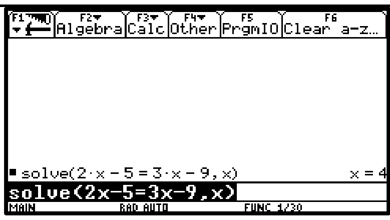
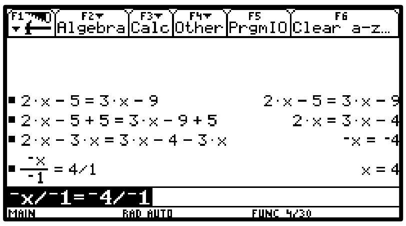
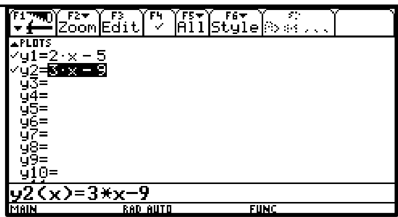
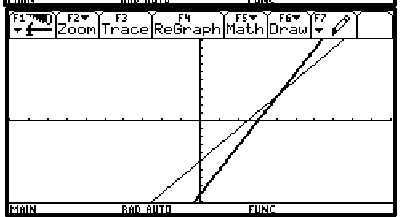
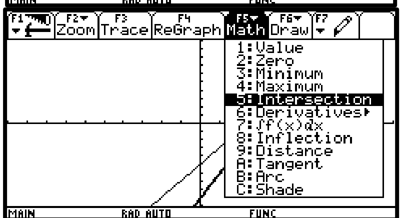
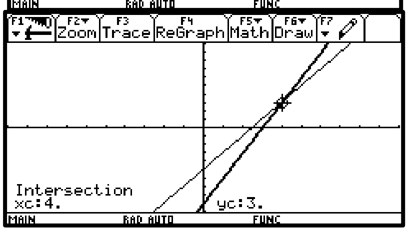
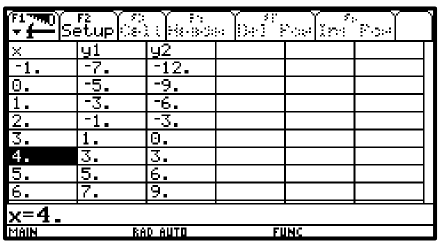
4. The Algebra Menu

Menu Item	**Description F2 MENU**  
1: solve	Solves an expression for a specified variable. This returns solutions only, regardless of the Complex Format mode setting (For complex solutions, select A:Complex from the algebra menu).
2: factor	Factorises an expression with respect to all its variables or with respect to only a specified variable.
3: expand	Expands an expression with respect to all its variables or with respect to only a specified variable.
4: zeros	Determines the values of a specified variable that make an expression equal to zero.
5: approx	Evaluates an expression using floating-point arithmetic, where possible.
6: comDenom	Calculates a common denominator for all terms in an expression and transforms the expression into a reduced ratio of a numerator and denominator.
7: propFrac	Returns an expression as a proper fraction expression.

Solving Linear Equations

Example. Solve $2x - 5 = 3x - 9$.

We can solve this in three different ways: algebraically, graphically, and numerically.

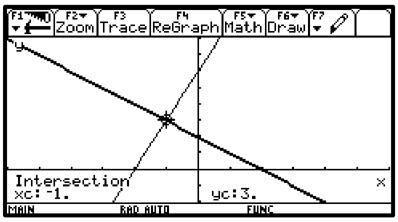
Press	See	Explanation
Method 1 a) [HOME] F2 1 2*x* -5 = 3*x* -9 ,*x*) [ENTER]		$2x - 5 = 3x - 9$ is solved by an algebraic method. The , *x* tells the calculator to solve with respect to *x*. $x = 4$ is the value which makes both sides equal in value.
Method 1 b) 2*x* - 5 = 3*x* - 9 [ENTER] 2*x*–5 +5 = 3*x*-9+5[ENTER] 2*x*-3*x* = 3*x* -3*x* [ENTER] ⁻*x*/⁻1 = ⁻4 / ⁻1[ENTER]		To find the value of *x*, we need to simplify the given expression step by step: If we add 5 to both sides, the expression is simplified to $2x=3x - 4$. If we subtract $3x$, the expression is simplified to $-x= -4$. If we divide by -1, finally we get $x = 4$
Method 2. ◆ [F1] 2*x* - 5 [ENTER] 3*x* - 9 [ENTER] ◆ [F3] [F5] **5** 1st curve? [ENTER] 2nd curve? [ENTER] Lower bound? 0 [ENTER] Upper bound? 6 [ENTER]		Here each side of the equation is defined as a function, using y1(*x*) and y2(*x*): y1(*x*)= $2x - 5$ y2(*x*)= $3x - 9$ Looking at the two graphs, we can see that they intersect at one point. To find the intersection point we need to use the function key [F5]. 1st curve means y1(*x*), 2nd curve means y2(*x*). The lower and upper bound means the interval in which the intersection point is found. So the two graphs intersect at the point (4, 3). i.e. $x = 4$
Method 3. ◆ [F4] tblStart: –1 ∆tbl: 1 ◆ [TABLE]		The point of intersection can be found using a table. Enter y1 and y2 as in method 2. When we look at the point *x*=4, we can see the values of the two functions are the same, and equal to 3.

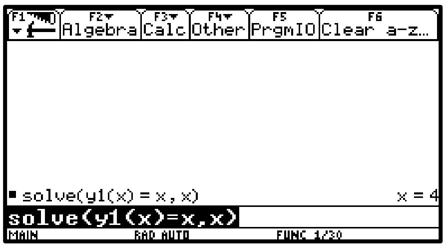
Looking at the three methods we see that the value of x is the same in each case.
Exercise
Solve the following equations. Make sure you use each of the three methods above at least once.

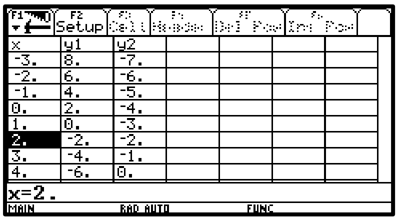
1. $|3x - 2| = 5$
2. $x^2 - 2x + 7 = 22$
3. $\sqrt{2 - x} = x$
4. $\ln\left(\frac{x+1}{2}\right) - \ln\left(\frac{x}{2}\right) = 3$
5. $e^{4x} = 4^{3-2x}$ (give the exact solution)

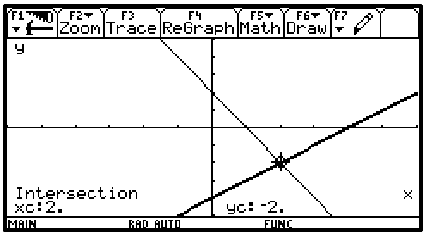
Screen-snaps Exercise

Reproduce the following screens on your TI-89.

1.


2.


3.




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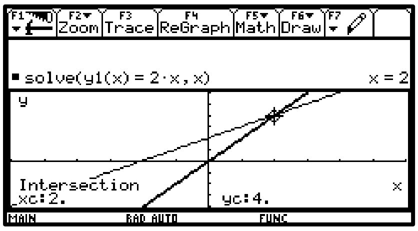


For this question you will need to use the split-screen facility using:
[F2] 'Split Screen' – see page 3.

Investigation

Find all the integer values of a for which $ax + 1 = 3x + 5$ has integer solutions.

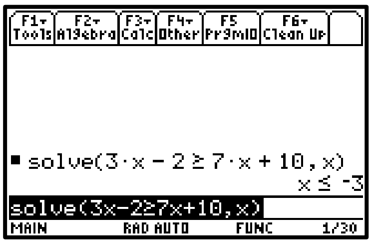
Inequalities

Example. Solve $3x - 2 = 7x + 10$

Method 1)

[F2] $3x$ - 2 [◆] >= $7x$ +10 , x) [ENTER]

Note: The >= automatically changes to ≥ once the equation is entered.



Let us now solve the inequality step by step.

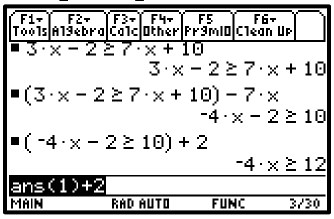
Method 2) In the following we transform in an equation into the form ‘x = or = …’ by specifying equivalent transformations.

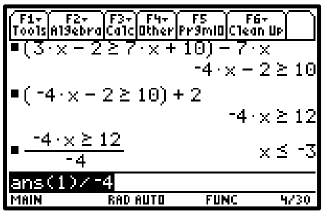
Step 1. $3x - 2 = 7x + 10$ [ENTER]

The subtraction of $7x$ is a reasonable first step.

Step 2. The application of the equivalent transformation of adding $-7x$ to both sides of the equation, adding 2 and dividing 4.

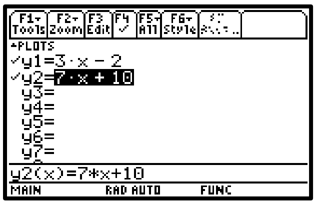
[2nd][ANS] - $7x$ [ENTER]   [2nd][ANS] + 2 [ENTER]   [2nd][ANS]/⁻4 [ENTER]

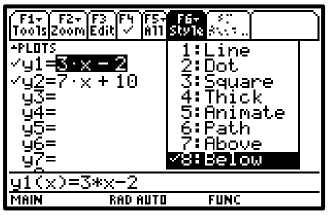


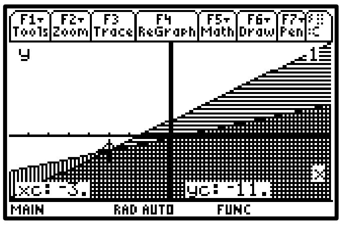


Note: ans(1) always contains the last answer, ans(2), ans(3), etc, also contain previous answers. For example, ans(2) contains the next to last answer.

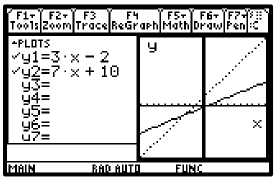
Method 3)

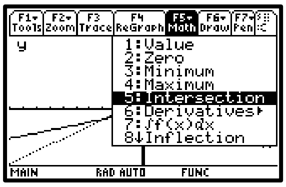


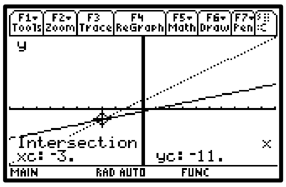


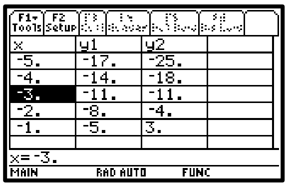


Method 4)









Exercise. Solve the following inequations:

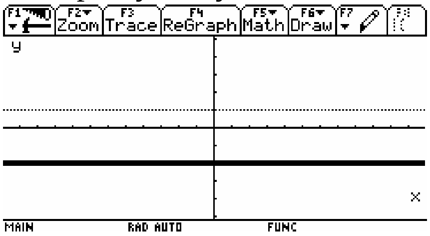
1. $|4x - 2| = 6$ 2. $|4x - 2| = 6$

5. Types of functions

Constant Function

$$f(x) = c$$

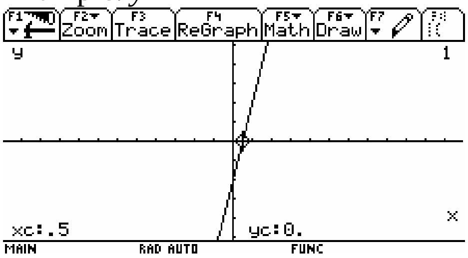
Example. $y = 1$, $y = -2$

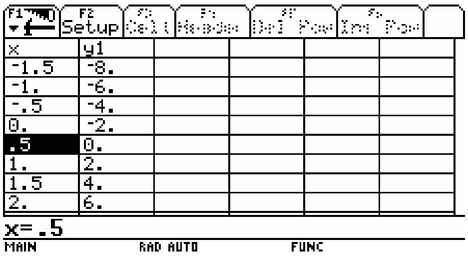


Linear Function

$$f(x) = mx + b$$

Example. $y = 4x-2$

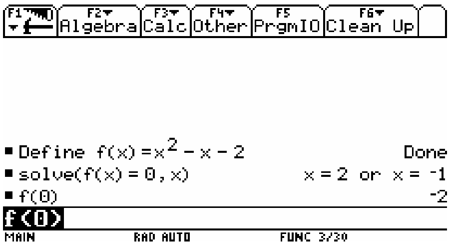


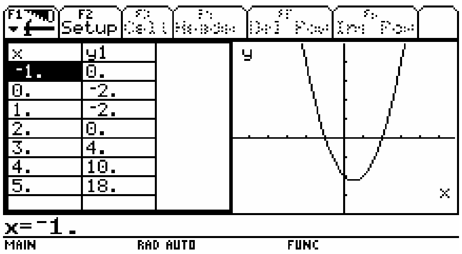


Quadratic Functions

$$f(x) = ax^2 + bx + c, \quad a ? 0$$

Example: $y = x^2 - x - 2 = (x-2)(x+1)$

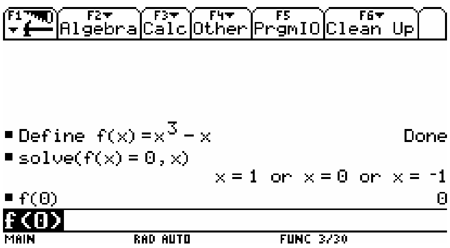


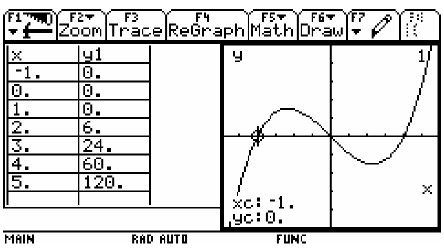


Cubic Functions

$$f(x) = ax^3 + bx^2 + cx + d, \quad a ? 0$$

Example. $y = x^3 - x = x(x+1)(x-1)$





Combinations of functions

$(f + g)(x) = f(x) + g(x)$

$(f - g)(x) = f(x) - g(x)$

$(f \bullet g)(x) = f(x) \bullet g(x)$

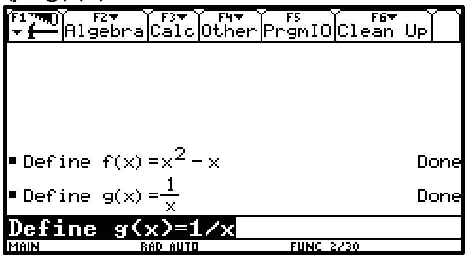
$(f / g)(x) = f(x) / g(x)$

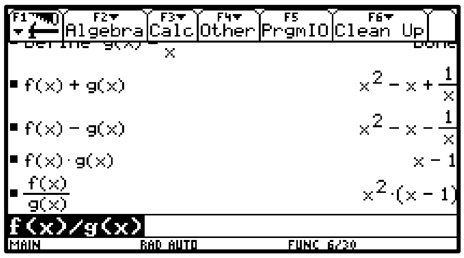
Example. Let $f(x) = x^2 - x$, $g(x) = \dfrac{1}{x}$

$(f + g)(x) =$

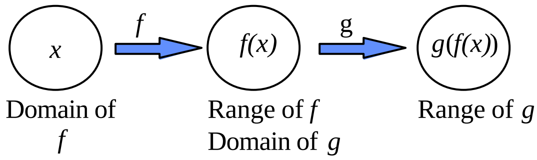
$(f - g)(x) =$

$(f \bullet g)(x) =$
$(f\,/\,g)(x) =$



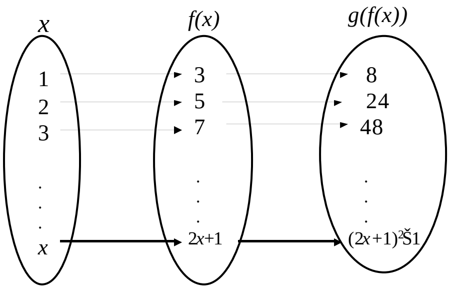


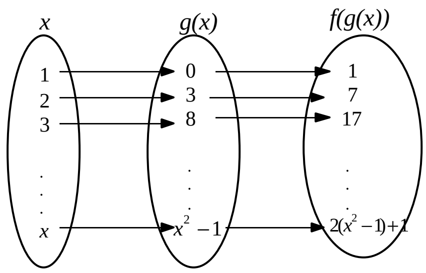
Composite Functions



Example. $f(x) = 2x + 1$ and $g(x) = x^2 - 1$

$(g \circ f)(x) = (2x + 1)^2 - 1 = 4x(x + 1)$ $\qquad$ $(f \circ g)(x) = 2(x^2 - 1) + 1 = 2x^2 - 1$





Definition $(f \circ g)(x) = f(g(x))$ $\qquad$ cf. inverse $f \circ f^{-1} = I$ $(g \circ f)(x) = g(f(x))$

f(g(x)) is a function of a function. The domain of ***f* o *g*** is the set of all numbers *x* in the domain of *g* such that *g(x)* is in the domain of *f*.

Example. $f(x) = \sqrt{2x - 3}$, $g(x) = x^2 - 1$

$(f \circ g)(x) =$

$(g \circ f)(x) =$

$(g \circ g)(x) =$

The Exponential Function

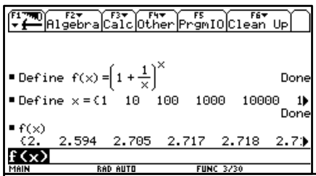
The exponential function is given by

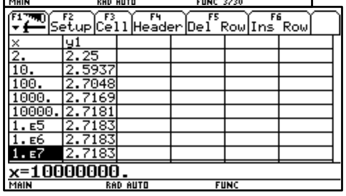
$$f(x) = e^x$$

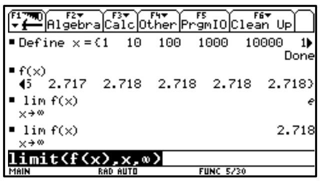
where the base *"e"* is approximately equal to 2.7182818284.

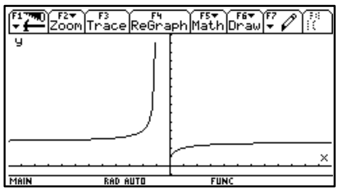
n	1	10	100	1000	10000	100000	1000000
$\left(1+\frac{1}{n}\right)^n$	2	2.594	2.705	2.717	2.718	2.718	2.718

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845904\ldots = e$$

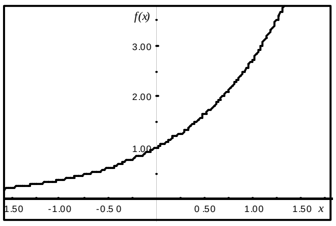




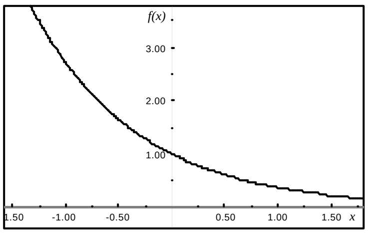




f(x) = e^x



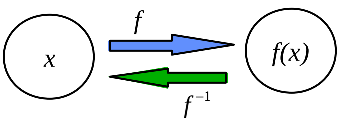
f(x) = e^-x



Domain: $x \in$ R
Range: $y > 0$, $y \in$ R

Domain: $x \in$ R
Range: $y > 0$, $y \in$ R

Inverse Functions



Definition.

Let f be a ***one – to – one function*** with domain A and range B.
Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

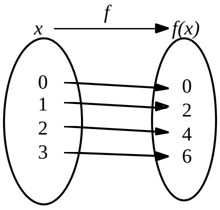
- Do not mistake the – 1 in f^{-1} for an exponent. Thus

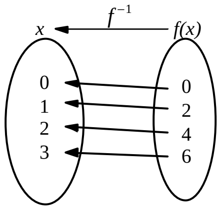
$f^{-1}(x)$? $\frac{1}{f(x)}$

(The reciprocal $\frac{1}{f(x)}$ could be written as $[f(x)]^{-1}$

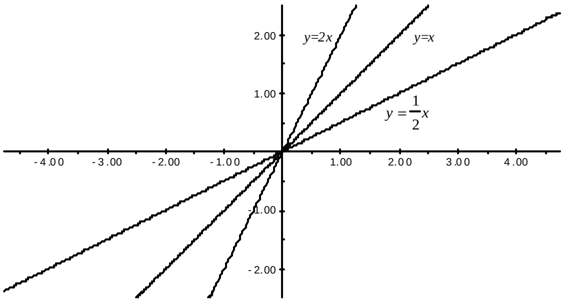
Example 1.
Find the inverse of the function f given by the following set:
f = {(10, 20) (15, 15) (25, 3) (27, 3)}
Answer: f^{-1} = {(20, 10) (15, 15) (3, 25) (3, 27)}

Example 2. Find the inverse of the function $y = 2x$





If the function is given as a graph, you must reflect the graph in the line $y = x$ to find the graph of the inverse.



How to find the inverse function.

Step 1. Write $y = f(x)$.

Step 2. Solve this equation for x in terms of y.

Step 3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example. Find the inverse function of $y = \sqrt{x}$.

Sol)

Step 1. Write $y = f(x)$.

$$y = \sqrt{x} \quad (x = 0, y = 0)$$

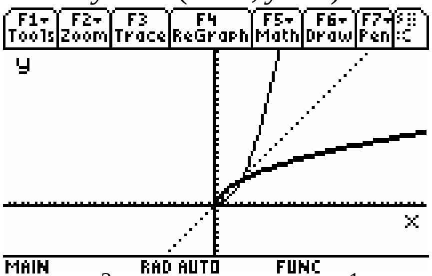
Step 2. Solve this equation for x in terms of y.

$$y^2 = x \; (x = 0, y = 0)$$

$$\text{so } x = y^2$$

Step 3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

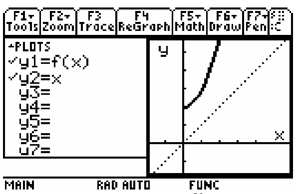
$$y = x^2 \; (x = 0, y = 0)$$

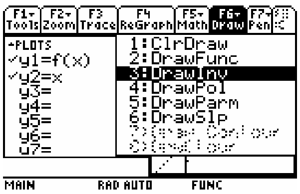


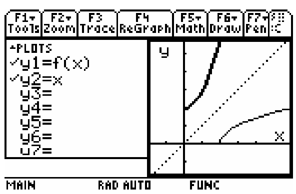
Example: For the given function, $f(x) = x^2 + 2$ ($x = 0$), find $f^{-1}(x)$, the inverse of f.

Solution: Since $x^2 = y - 2$, $x = \sqrt{y-2}$ ($y = 2$)

The inverse function is $y = \sqrt{x-2}$ ($x = 2$)

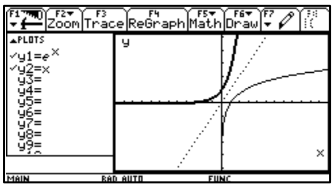






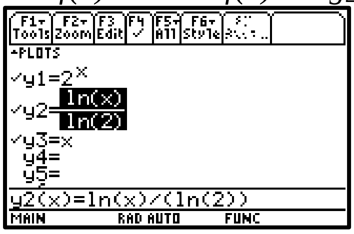
Example. $y = e^x$ and $y = \ln x$

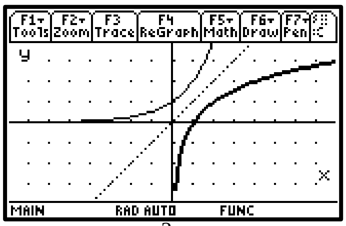
(since $x = \ln y$)



Logarithmic & Exponential Functions

Logarithmic functions are the **inverse functions** to exponential functions.
Let $f(x) = 2^x$ and $f(x) = log_2 x$ are a pair of inverse functions.





$$2^3 = 8 \Leftrightarrow 3 = \log_2 8$$
$$10^4 = 10000 \Leftrightarrow 4 = \log_{10} 10000$$
$$10^{0.4771} = 3 \Leftrightarrow 0.4771 = \log_{10} 3$$
$$3^n = 243 \Leftrightarrow n = \log_3 243$$

$\boldsymbol{a^n = x}$ Û $\boldsymbol{n = \log_a x}$ **(*a* > 0, *a* ? 0)**

(the logarithm of x to base a is said to be n)
Logarithms using base e are called natural logarithms, and $\log_e x = \ln x$

Rational functions

An asymptote is the *behaviour of a function* (or the graph of a function) *for extremely large values of x or y*. For *very large values of x or y,* graph of $y = f(x)$ gets close to the asymptote.

Rational functions are of the form : $f(x) = \dfrac{p(x)}{q(x)}$

(where *p(x)* and *q(x)* are polynomial expressions *q(x)* ? 0)

<u>Asymptotes:</u>

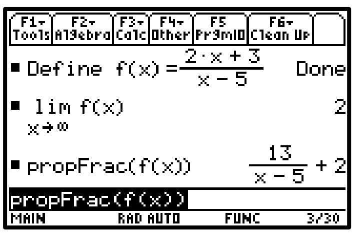
$$f(x) = \frac{ax+b}{cx+d} = \frac{k}{x-p} + q$$

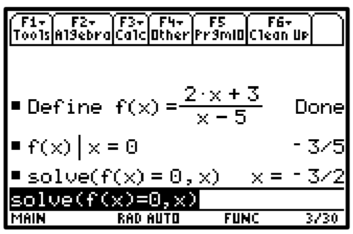
(To make simplify divide each term in the numerator and denominator by the highest power of x which appears)

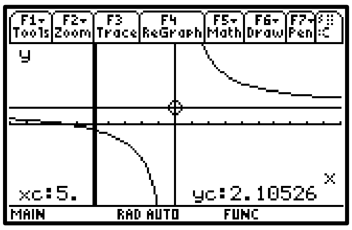
- **Vertical asymptote**: $\boldsymbol{x = p}$, (this is found by equating the *denominator to zero* and solving the resulting equation.)
- **Horizontal asymptote**: $\boldsymbol{y = q}$, (this is found by finding the *limit of the function* as x gets very large.)
- Find the y-intercept by substituting $x = 0$ in the function.
- Find the x-intercept by equating $f(x) = 0$, and solving for x.
- Domain: x ? p, $x \in$ R
- Range: y ? q, y $\in$ R

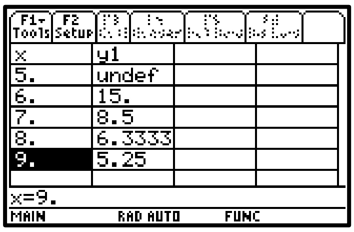
Example. Sketch the function $f(x) = \dfrac{2x+3}{x-5}$, identifying all intercepts with the axes and all asymptotes.

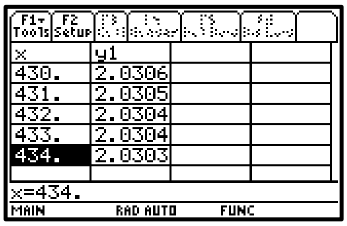
$$f(x) = \frac{2x+3}{x-5} = \frac{13}{x-5} + 2$$











Vertical asymptote:

Horizontal asymptote:

y-intercept:

x-intercept:

Domain:

Range:

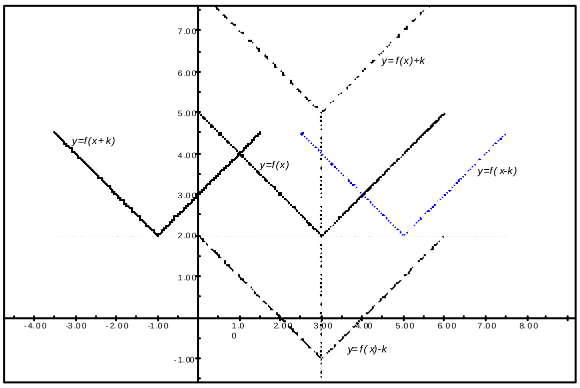
Exercise.

1. Let $f(x) = \dfrac{x^2 - x - 6}{x + 2}$. Sketch the graph of *f(x)* including any *x* and *y* intercepts. Can you explain why the graph has this form?

2. Given $g(x) = \dfrac{2x+3}{x-5}$ is invertible on *x*? 5, find $f^{-1}(x)$, the inverse of *f*.

6. Transformations

$y = f(x) + k$	K units upward
$y = f(x) - k$	K units downward
$y = kf(x)$	Vertically by a factor of k
$y = -f(x)$	Reflect the graph of $y = f(x)$ in the x axis
$y = f(-x)$	Reflect the graph of $y = f(x)$ in the y axis
$y = f(x - m)$	Shift the graph of $y=f(x)$, m units to the right
$y = f(x + m)$	Shift the graph of $y=f(x)$, m units to the left



Transformations parallel to the x– and y–axis

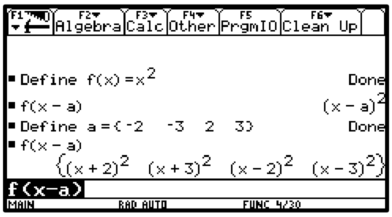
Example. The purpose here is to explain the relationship between $f(x)$, $f(x-a)$ and $f(x)+b$. Define the function $f(x)=x^2$.

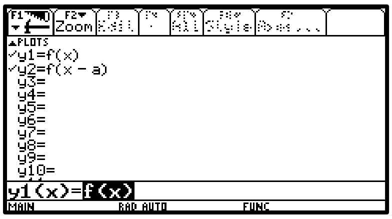
If $f(x)$ is defined by x^2 then $f(x - a)$ and $f(x) + b$ are found to be $(x-a)^2$ and $x^2 + b$ by following these instructions:

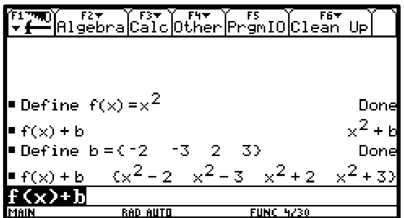
[HOME] [F4] 1 $f(x)=x$ ^2 [ENTER]
$f(x - a)$ [ENTER]
[F4] 1 $a = \{-2, -3, 2, 3\}$ [ENTER]

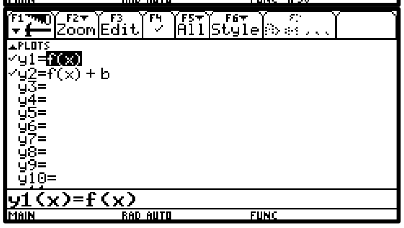
[HOME] [F4] 1 $f(x)=x$ ^2 [ENTER]
$f(x) + b$ [ENTER]
[F4] 1 $b = \{-2, -3, 2, 3\}$ [ENTER]

The bracket allows us to be enter a number of different values for a or b.



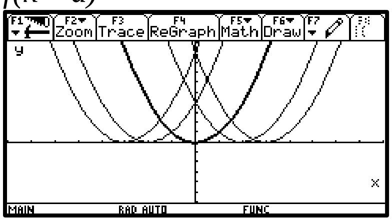




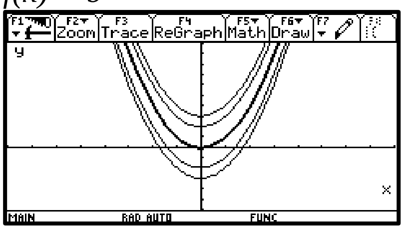


Note that when drawing $f(x - a)$ and $f(x) + b$ the calculator uses each of the values of a or b entered, showing the effect of them. We can see that the effect of

$f(x - a)$



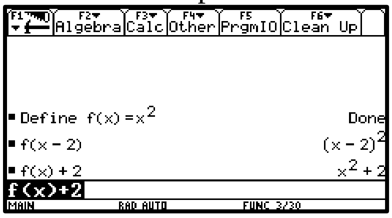
$f(x) + b$

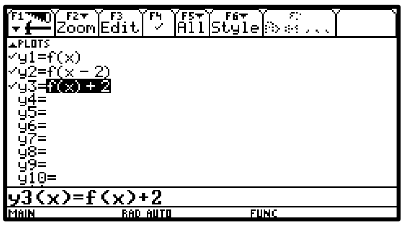


We can also deal with single values.

e.g. Compare the general functions $f(x-2)$ and $f(x) + 2$ for $f(x) = x^2$.

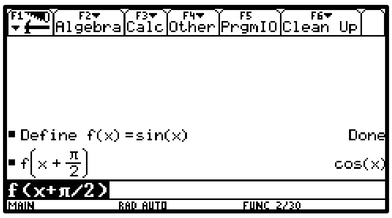
Looking at the general functions $f(x–2)$ and $f(x) + 2$ we can see that those functions correspond to the actual functions $(x–2)^2$ and $x^2 + 2$ based on the function $f(x) = x^2$, and the graph shows the transformation parallel to the x- and y-axes.

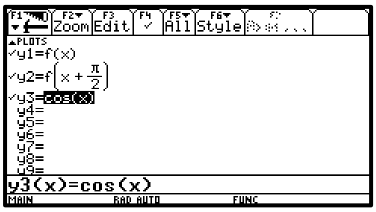


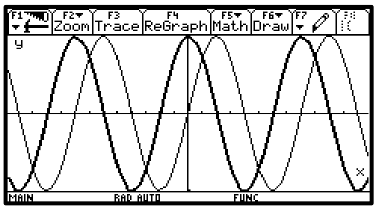


Sine and Cosine Function

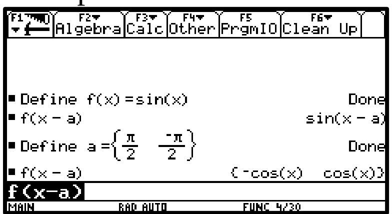
Example. Show that $\sin\left(x+\frac{p}{2}\right) = \cos x$

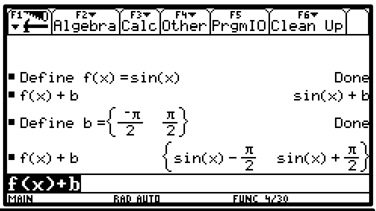


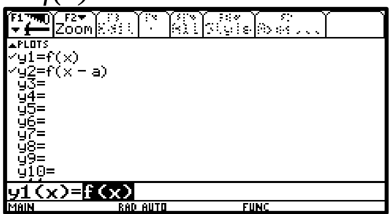


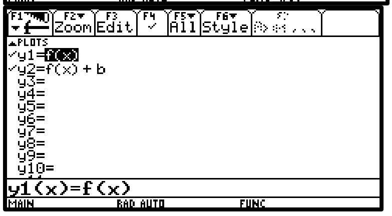


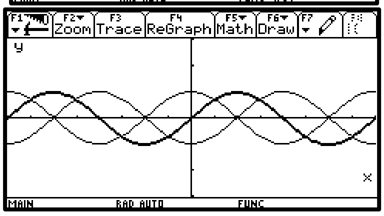
Example. Find the difference between $f(x–a)$ and $f(x) + b$ when $f(x) = \sin x$.

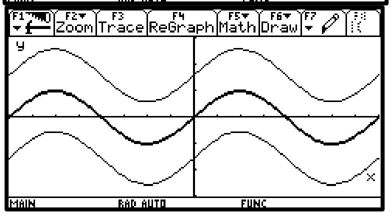












7. Limits

- **The Calc Menu**

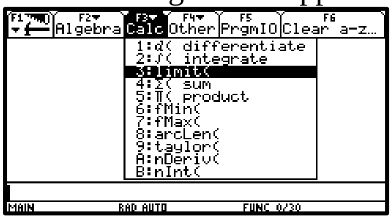
From the Home screen, press [**F3**].

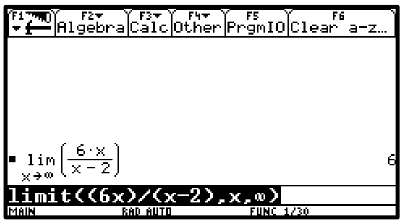
Menu Item	**Description**
d differentiate	Differentiates an expression with respect to a specified variable
$\int$ integrate	Integrates an expression with respect to a specified variable.
limit	Calculates the limit of an expression with respect to a specified variable

To find $\lim\limits_{x\to\infty} \dfrac{6x}{x-2}$ follow the key sequence.

[F3] 3 (6 *x*)/(*x* - 2), *x* , [◆] [CATALOG]) [ENTER]

The following should appear on your calculator screen.





Note: Put both numerator and denominator in brackets.

Example 1. Find $\lim\limits_{x\to 0} x\cos x$

We can get a sense of the limit by defining the function as *f(x)* and getting values of *x* near to zero.

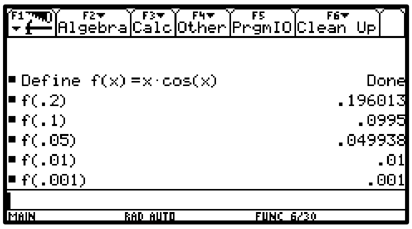
To find $\lim\limits_{x\to 0} x\cos x$ follow the key sequence:

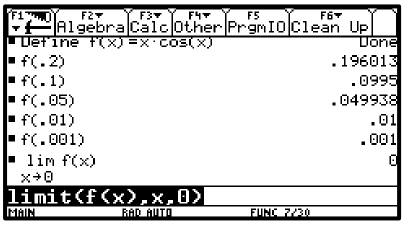
[F4] 1 *f(x)* = *x* cos(*x*) [ENTER]

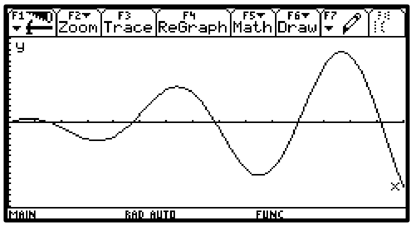
Whenever we change *x* taking steps of *x* closer to 0 then the value of *f*(*x*) is getting closer to 0.

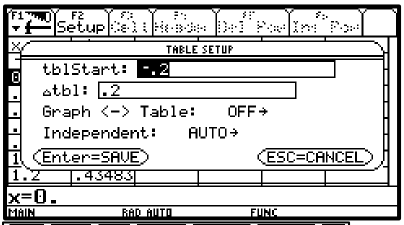
[F3] 3 *f(x)* , *x* ,0) [ENTER]

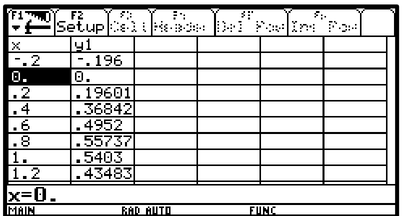
We can confirm our guess by asking for the limit.

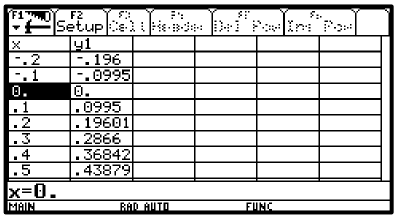


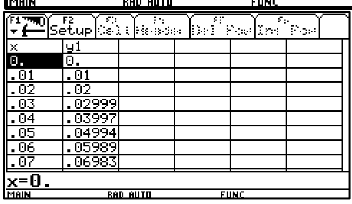


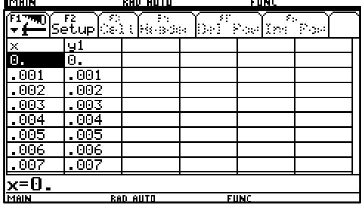












The graph and table help to confirm, in other representations, that the function has a limit of zero when $x\to 0$.

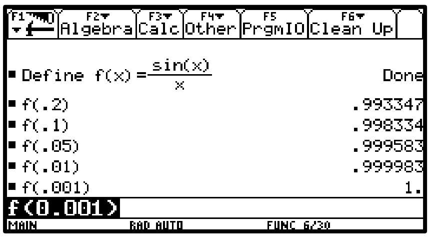
Example 2. Find $\lim_{x \to 0} \dfrac{\sin x}{x}$

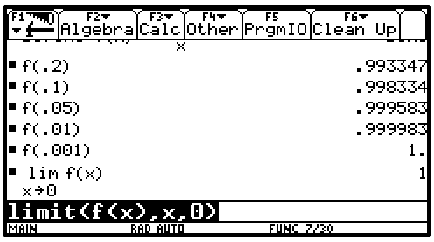
This is an important limit, but one that cannot be found by putting $x = 0$, since the function is undefined for $x = 0$.

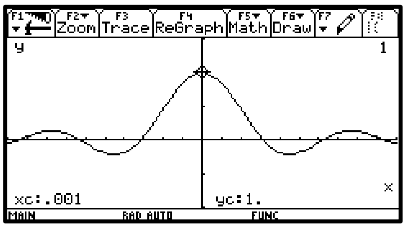
[F4] 1 $f(x) = \sin(x)$ [÷] x) [ENTER]

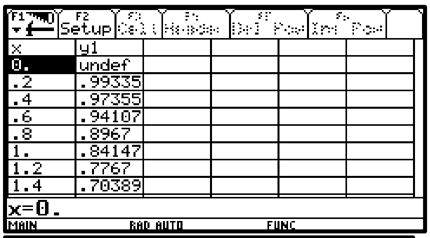
Whenever we change x taking steps of x closer to 0 then the value of $f(x)$ is getting closer to 1.

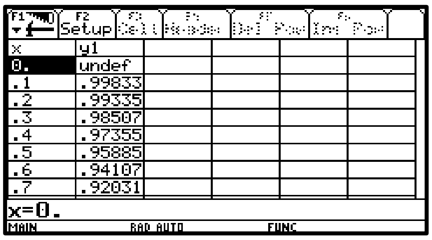
[F3] 3 $f(x), x, 0$) [ENTER]

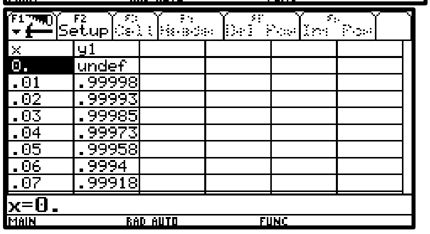


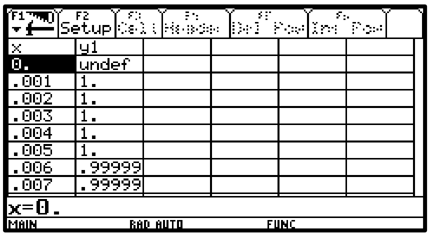












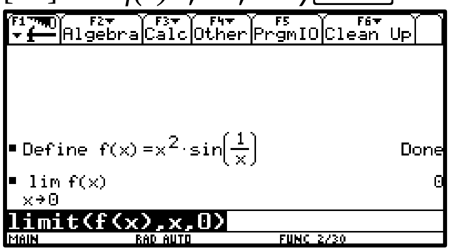
Again the graph and table provide supporting evidence for the limit.

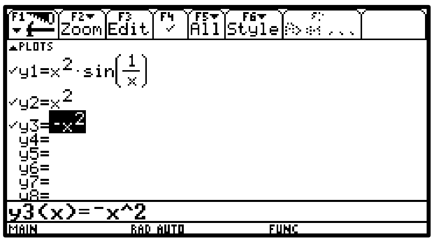
Example 3. $\lim_{x \to 0} x^2 \sin \dfrac{1}{x}$

[F4] 1 $f(x) = x^2 \sin(x)$ [ENTER]

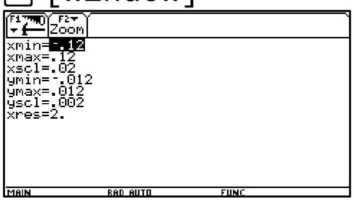
[F3] 3 $f(x)$, x , 0) [ENTER]

[◆] [y=]

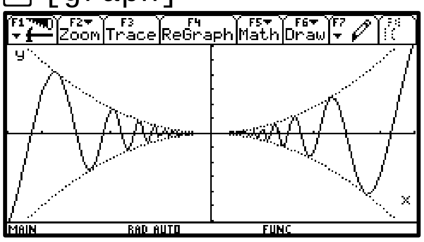




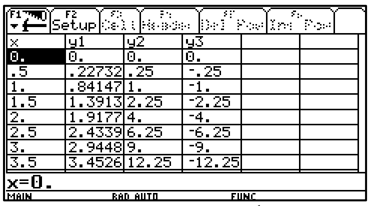
[◆] [window]

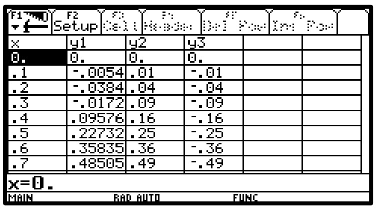


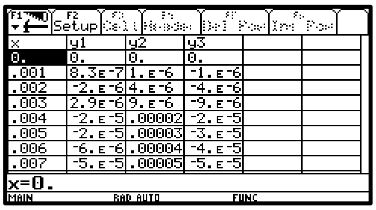
[◆] [graph]



◆ [TABLE]

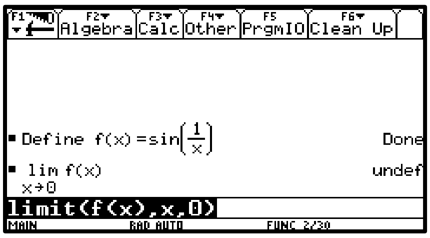


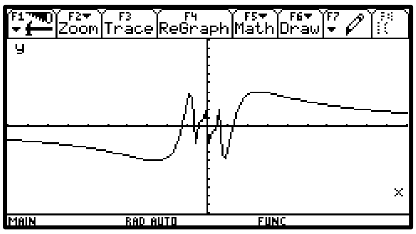


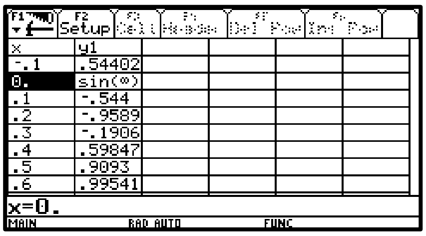


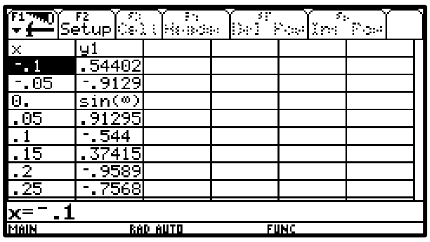
Example 4. $\lim_{x\to 0} \sin\frac{1}{x}$

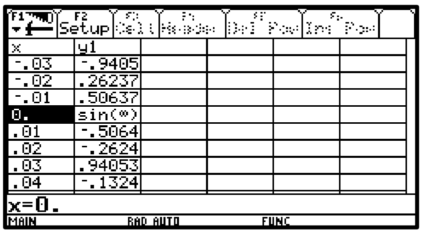
Some limits do not exist. We can build an understanding of the reasons for this.











We can plot the graph and zoom in on x = 0 or from the table we can see that no matter how much we zoom in on x = 0 values either side are the same but differ in sign. This leads to the idea of left and right limits.

Left and Right Limits and Differential Functions

We can use the left and right limits to see why some functions are not differentiable at certain points.

Consider the expression

$$f(t) = \frac{t^2 - 7t + 10}{t - 2}$$

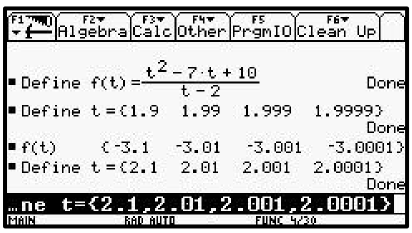
Define the function: [F4] 1 *f(t)* = (t^2 – 7t + 10) ÷ (t – 2) ENTER

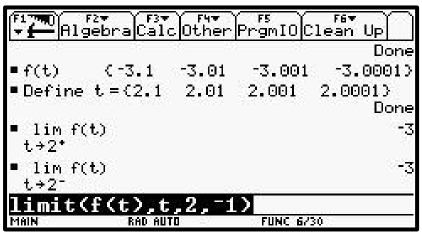
Investigate right limit: [F4] `1 t = {1.9, 1.99, 1.999, 1.9999}` then evaluate $f(t)$

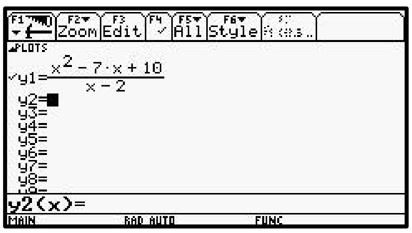
Investigate left limit: [F4] `1 t = {2.1, 2.01, 2.001, 2.0001}` then evaluate $f(t)$

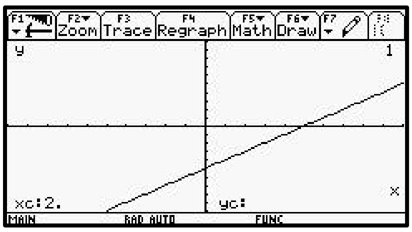
Right limit is: [F3] 3 $f(t)$, t, 2, -1) ENTER

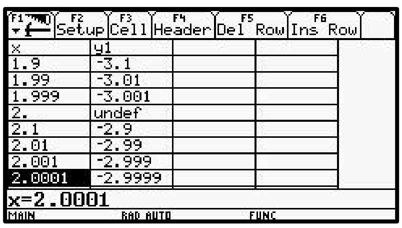
Left limit is: [F3] 3 $f(t)$, t, 2, 1) ENTER











Example 5. Find $\lim_{x \to 2} f(x)$ for the function $f(x) = \begin{cases} x^2 & \text{for } x < 2 \\ 6 - x & \text{for } x \ge 2 \end{cases}$.

Define the piecewise functions by using the following instructions.

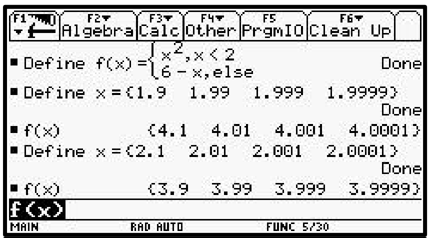
[F4] 1 $f(x)$ = when (x^2 , x < 2, 6 – x) [ENTER]

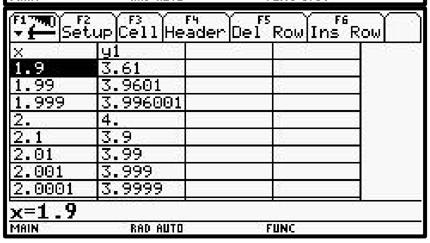
Investigate right limit: [F4] 1 x = {1.9, 1.99, 1.999, 1.9999} then evaluate $f(x)$

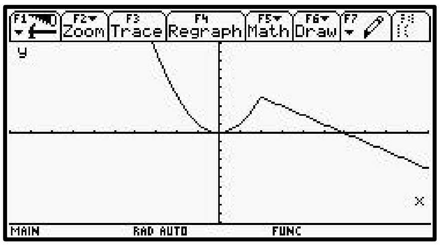
Investigate left limit: [F4] 1 x = {2.1, 2.01, 2.001, 2.0001} then evaluate $f(x)$

Right limit is: [F3] 3 $f(x)$, x , 2, -1) [ENTER]

Left limit is: [F3] 3 $f(x)$, x , 2, 1) [ENTER]







Exercise.

Using the symbolic, graphical and tabular representations find these limits if possible.

1. $\lim_{x \to 2} (3x - 1)$

2. $\lim_{x \to 2} \dfrac{x^2 + 5x - 14}{x^2 - x - 2}$

3. $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$

4. $f(x) = \begin{cases} x & (x \le 0) \\ x^2 - 2x - 3 & (x \ge 0) \end{cases}$, $\lim_{x \to 0} f(x)$

5. $\lim_{x \to 0} |x|$

6. $\lim_{x \to -3^-} \dfrac{\sqrt{x^2 - 9}}{x + 3}$

Techniques for finding limits

(a) Numerically (substitute numbers from both sides)

(b) Direct substitution

(c) Algebraic Cancellation then substitution

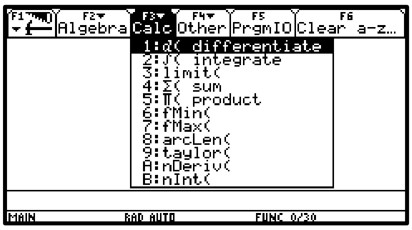
(d) Limits as $x \to \infty$ (divide top and bottom by the highest power of x)

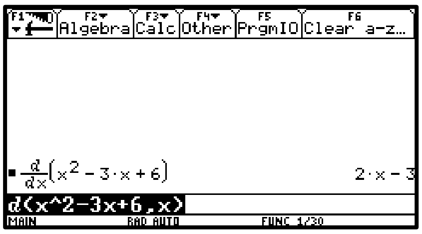
Summary table for common cases if you substitute first:

Result when substituting	**Conclusion**
Sensible answer	This is the limit
$\dfrac{number \ne 0}{0}$	Limit does not exist
$\dfrac{0}{number \ne 0}$	Limit = 0
$\dfrac{0}{0}$	Factorise, cancel, and try again

8. Differentiation

We can get the calculator to differentiate directly and give the answer:
To differentiate the function $y = x^2 - 3x + 6$ follow these key sequence instructions:
[F3] 1 $x^2 - 3x + 6, x$) [ENTER]
The following should appear on the calculator display.





Notes:
We type **comma *x*** at the end of the expression because we are differentiating with respect to ***x***.
There is no need to type in the multiplication sign between 3 and ***x***.
All expressions are enclosed in brackets.

Exercise
Find the derivative of each of the following functions using the TI–89

1. $x^2 + 5x^3$
2. $20x^8 + 9x^3 + 52$
3. $(x-6)(x+5)$
4. $\dfrac{x^2 - 9}{x + 3}$
5. $\dfrac{e^{-2x}}{3e^x - 1}$
6. $2x^3 \sin^2 x - \cos(2x - 1)$
7. $3x^2 \ln x$
8. $\dfrac{3}{x^2}$
9. $\sqrt{9x^2 - 36}$

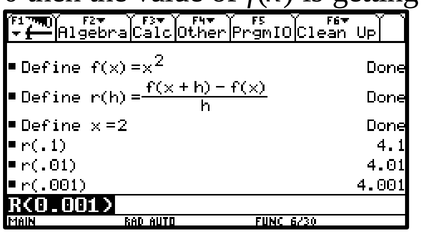
Answers:

1. $15x^2 + 2x$
2. $27x^2 + 160x^7$
3. $2x - 1$
4. 1
5. $\dfrac{-(9e^x - 2)e^{-2x}}{(3e^x - 1)^2}$
6. $2\sin(2x-1) + 4x^3 \sin(x)\cos(x) + 6x^2(\sin(x))^2$
7. $6x\ln(x) + 3x$
8. $\dfrac{-6}{x^3}$
9. $\dfrac{3x}{\sqrt{x^2 - 4}}$

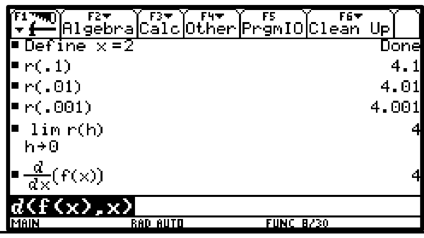
Example 1. Find the derivative of $f(x) = x^2$ at $x = 2$.
We can do differentiation from first principles by using the ideas of limits we have developed.

Method 1.

[F4] 1 $f(x) = x^2$ [ENTER]
In this method we use the calculator function *r(h)* (i.e. rate of change) at the point $x = 2$:
Whenever we change *h* taking steps of *h* closer to 0 then the value of $f(x)$ is getting closer to 4.

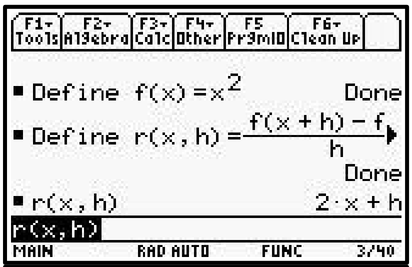


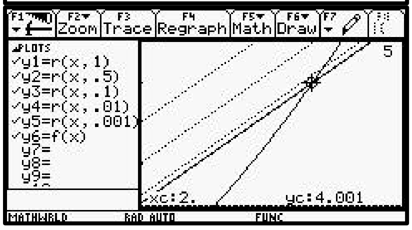
[F3] 3 *r(h)* , *h*, *h*, 0) [|] $x = 2$ [ENTER]
[F3] 1 *f(x)*, *x*) [|] $x = 2$ [ENTER]
We can confirm our guess by asking for the limit and differentiation.

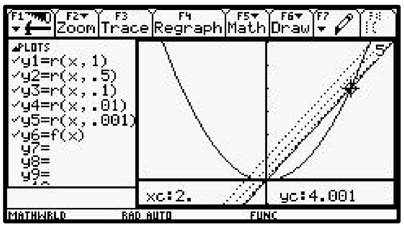


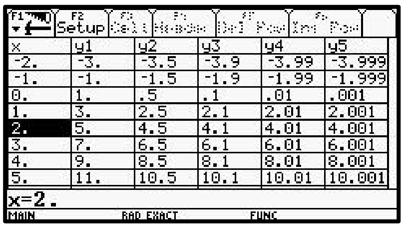
Thus the rate of change at $x = 2$: $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = f'(2) = 4$.

Method 2.
In this method we use the calculator function *r(x, h)* (i.e. rate of change) at the point $x = 2$:
Whenever we change *h* taking steps of *h* closer to 0 then the value of $f(x)$ is getting closer to 4.





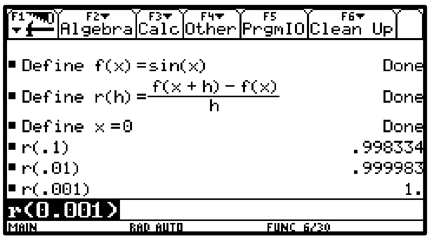


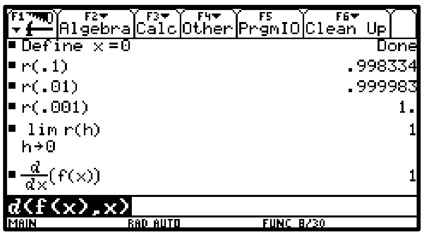


We can see the general function 2*x*.

Example 2. Find the derivative of *f(x)* = sin*x*

Method 1.

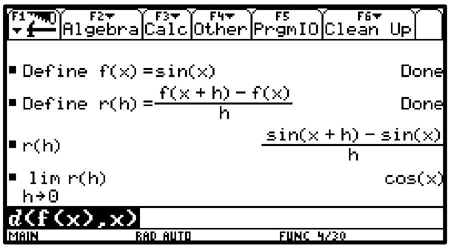


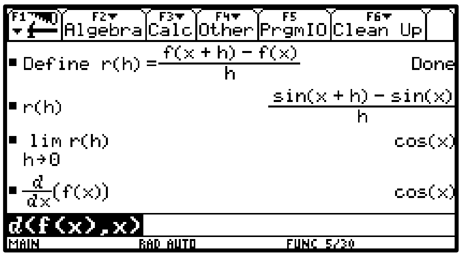


$$\frac{dy}{dx} = f'(0) = \lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h} = \cos(0) = 1$$

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \cos(x)$$

Method 2.

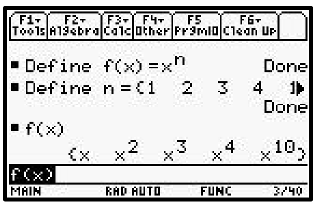


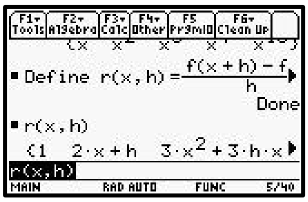


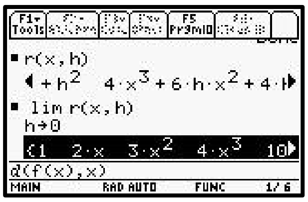
Example. Find the derivative of *f(x)* = x^n

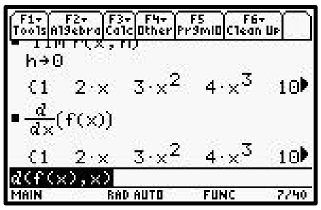
This example can be difficult from first principles if students do not have access to the binomial theorem.

Define the function $f(x) = x^n$. When we define the value of power, n = 1, 2, 3, 4, 10 the functions are changed to the actual functions, x, x^2, x^3, x^4, x^{10}. If we define the slope function *slope(h)* as the average rate of changed, then we can see that the derivative of the functions are 1, $2x$, $3x^2$, $4x^3$, $10x^9$ as follows:

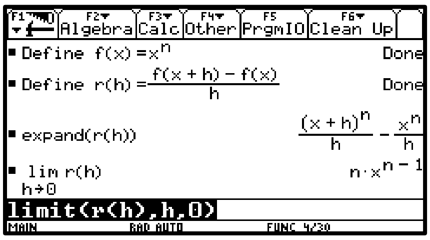


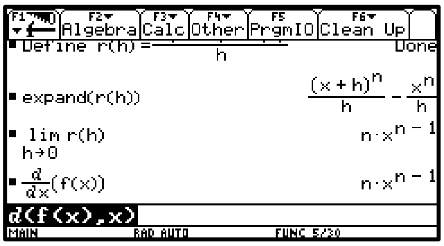






Defining the rate of function *r(h)*, we can get that the general derivative of x^n is nx^{n-1} as follows:





Thus $\dfrac{dy}{dx} = f'(x) = \lim\limits_{h\to 0}\dfrac{(x+h)^n - x^n}{h} = nx^{n-1}$

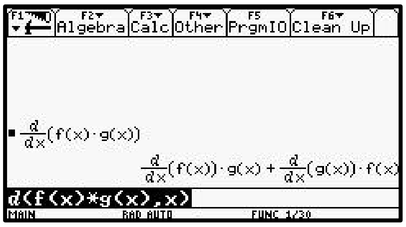
Differentiation Formulas

$f(x)$	f¢(x)
a) $f(x) = c$ (c is constant)	$f'(x) = 0$
b) $y = x^n$	$y' = nx^{n-1}$
c) $y = c \cdot f(x)$ (c is constant)	$y' = c \cdot f'(x)$
d) $y = f(x) + g(x)$	$y' = f'(x) + g'(x)$

Product rule

Where $y = u \cdot v$ and u and v are both functions of x, then:

$$\frac{dy}{dx} = \frac{du}{dx}\cdot v + u\cdot\frac{dv}{dx}$$

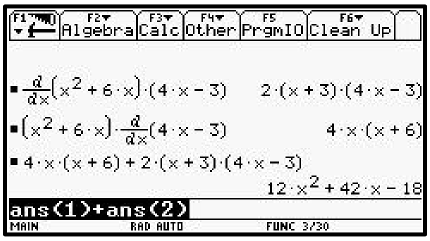


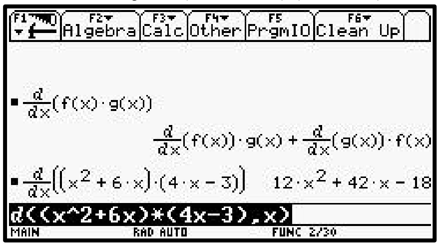
or

$y = f(x)g(x)$

$y' = f'(x)g(x) + f(x)g'(x)$

Example. Find the derivative of the function $y = (x^2 + 6x)(4x - 3)$





Exercise. Differentiate the following using the product rule.

1. $y = (x^3 - 3)(x^2 + 2)$

2. $y = \dfrac{2x^2 + 1}{x^2}$

3. $y = x\sqrt[3]{x}$

4. $y = \sqrt{x}(3x^2 - 1)$

Quotient rule

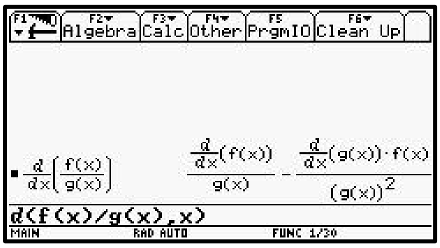
Where $y = \dfrac{u}{v}$ and u and v are both functions of x then

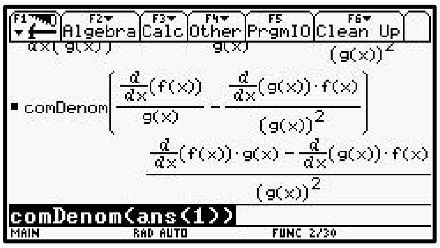
$$\frac{dy}{dx}=\frac{\frac{du}{dx}\cdot v-u\cdot\frac{dv}{dx}}{v^2}$$

or

$y=\frac{f(x)}{g(x)}$ $(g(x)\ ?\ 0)$

$$y'=\frac{f'(x)g(x)-f(x)g'(x)}{\{g(x)\}^2}$$

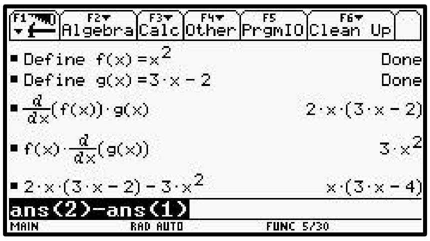


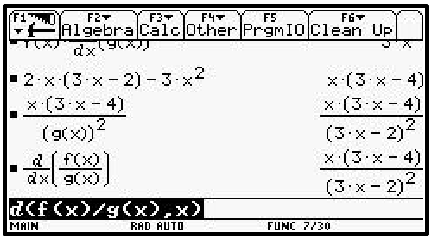


pf)

$$y=\frac{f(x)}{g(x)}=f(x)\cdot g(x)^{-1}$$
$$y'=f'(x)\cdot g(x)^{-1}-f(x)\cdot g(x)^{-2}\cdot g'(x)$$
$$=\frac{f'(x)}{g(x)}-\frac{f(x)\cdot g'(x)}{g(x)^2}$$
$$=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g(x)^2}$$

Example. If $y=\frac{x^2}{3x-2}$





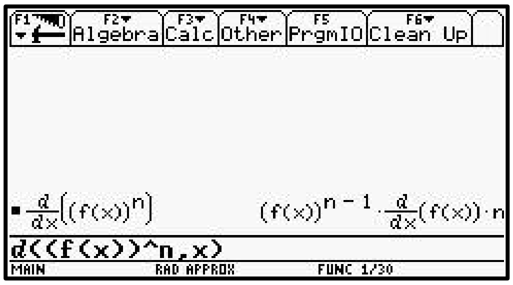
Exercise. Differentiate the following using the quotient rule.

1. $y=\frac{x-1}{x+3}$

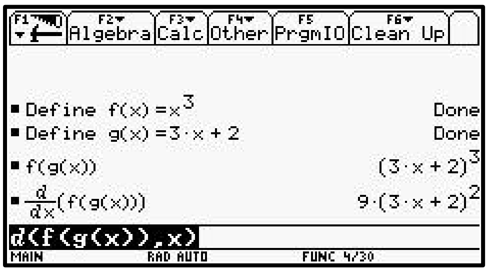
2. $y=\frac{x^2-6x}{x-3}$

3. $y=\frac{x-x^3}{\sqrt{x}}$

Chain Rule

Composite Function	
If $y=f(u),\ u=g(x)$ then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ or $f'(g(x))\cdot g'(x)$	
$y=f(ax+b)$ $\quad\frac{dy}{dx}=af'(ax+b)$ $y=\{f(x)\}^n$ $\quad\frac{dy}{dx}=n\{f(x)\}^{n-1}f'(x)$	

Example. If $y = (3x+2)^3$ find $\frac{dy}{dx}$



Exercise. Differentiate each of these.

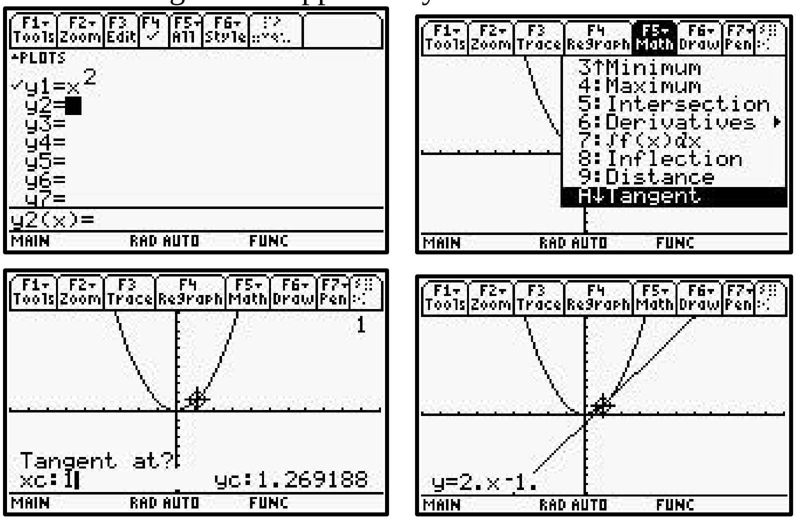
1. $y = (7x+5)^5$
2. $y = (4x^2 + 2x)^3$
3. $y = (7x - x^2)^{-2}$
4. $y = (ax + b)^3$
5. $y = (\frac{3}{5}x - \frac{1}{2})^2$
6. $y = \frac{4}{\sqrt{x+3}}$
7. $y = \frac{1}{1+\sqrt{x}}$
8. $y = (x+\sqrt{x})^4$

9. Finding the tangent line at a point on curve

To Find the equation of the tangent to $y = x^2$ at x=1

◆ [y=] x^2 ENTER ◆ [Graph] [F2] - 4 [F5] **A:Tangent**

The following should appear on your calculator:



Exercise.

Find the tangent line to a curve.

1. $2x^2 - x - 15$, at x= –1
2. $2x^3 - 4x^2 - 6x$, at x= –2
3. $2x^4 - 6x^3 - 2x^2 + 6x$, at x=3
4. $-2x^4 + x^3 + 17x^2 - x - 15$, at x=3

Answers:

1. y= –5x–17
2. y= 34x+48
3. y= 48x–144
4. y= –88x+264

The Increasing/Decreasing Concept

The increasing/decreasing concept can be associated with the slope of the tangent line.

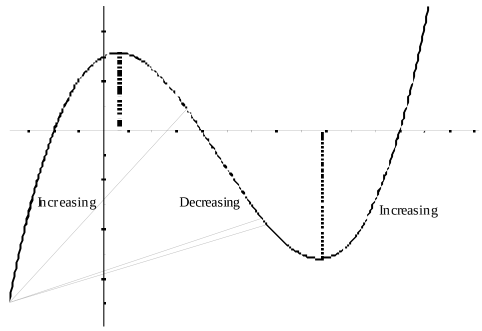
1. At a point (at which f is defined)

(j) If $f'(a) > 0$, then f is increasing at $x = a$

(k) If $f'(a) < 0$, then f is decreasing at $x = a$

2. On an interval (on which f is defined)

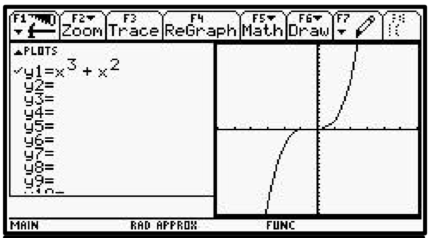
1. If $f'(a) > 0$ for all x in an interval, then f is increasing on the interval.

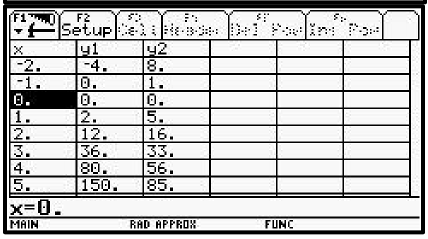


2. If $f'(a) < 0$ for all x in an interval, then f is decreasing on the interval.

Example. If $f(x) = x^3 + x^2$, is increasing or decreasing at x = 5?

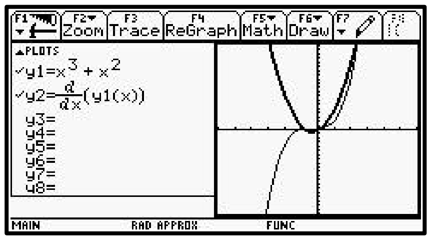
Find the intervals on which $f(x)$ is increasing or decreasing?





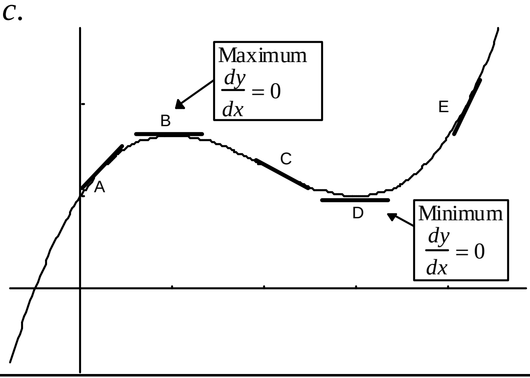
x	y1	y2
-2.	-4.	8.
-1.	0.	1.
0.	0.	0.
1.	2.	5.
2.	12.	16.
3.	36.	33.
4.	80.	56.
5.	150.	85.

x=0.



First Derivative Test

- Let c be a critical number of f and let f be continuous on an interval containing c. Then $(c, f(c))$ is a *relative maximum point* provided that $f'(x) > 0$ is an interval to the left of c and $f'(x) < 0$ in an interval to the right of c.
- Let c be a critical number of f and let f be continuous on an interval containing c. Then $(c, f(c))$ is a *relative minimum point* provided that $f'(x) < 0$ is an interval to the left of c and $f'(x) > 0$ in an interval to the right of c.



To find Maximum & Minimum Values:

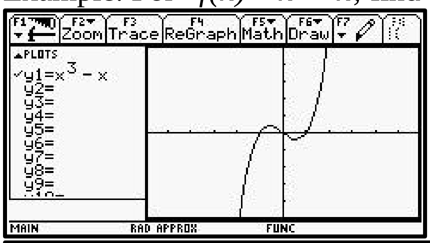
1) Find critical points ($f'(x)=0$) $\rightarrow$ x value

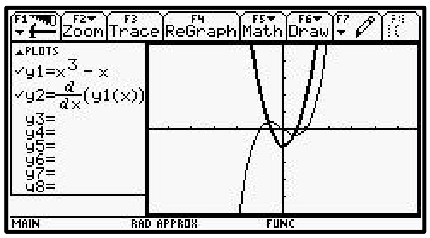
2) Substitute into f(x)

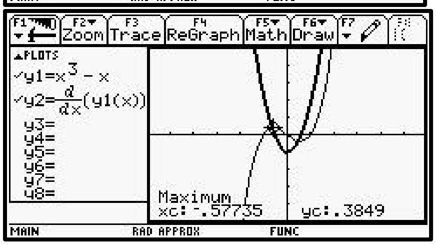
If $f'(x)$ changes $+ \rightarrow -$ Maximum value

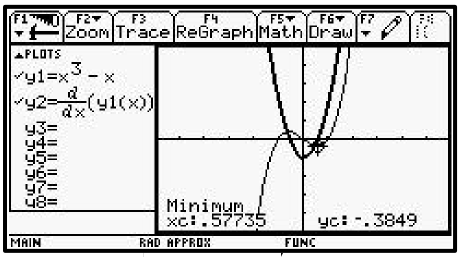
$- \rightarrow +$ Minimum value

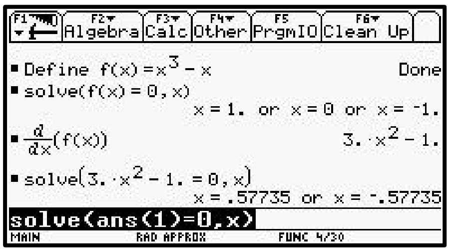
Example. For $f(x) = x^3 - x$, find maximum and minimum values.

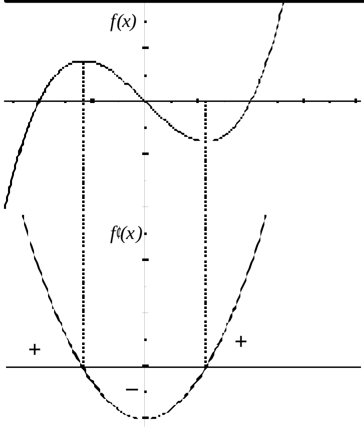




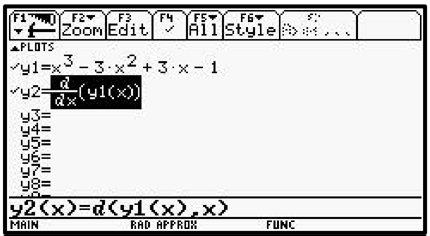


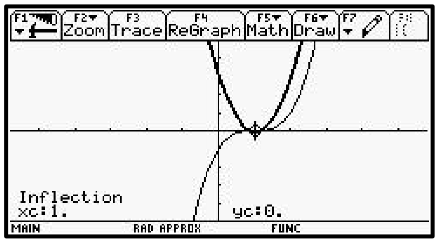


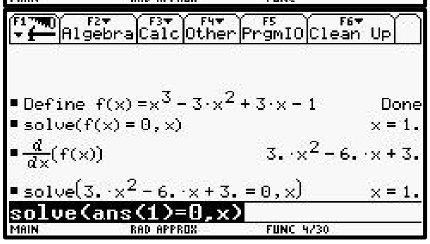




(e) The derivative can be zero without there being a relative maximum or relative minimum.
Example. $f(x) = x^3 - 3x^2 + 3x - 1$

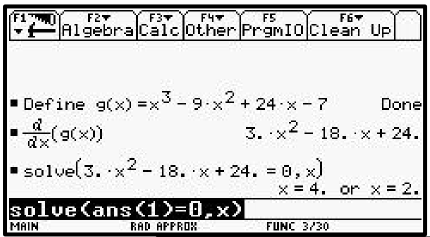


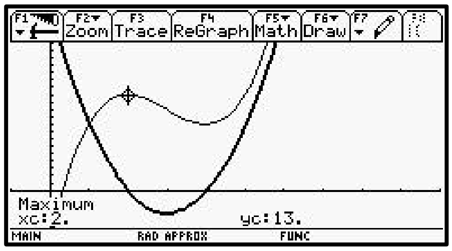


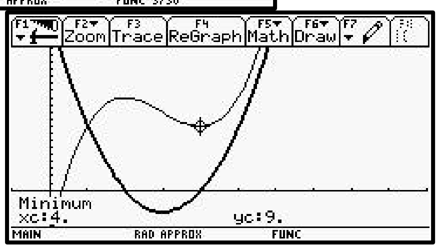


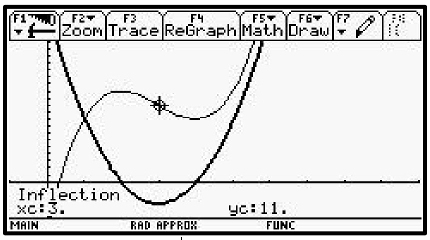
Local Maxima and Minima

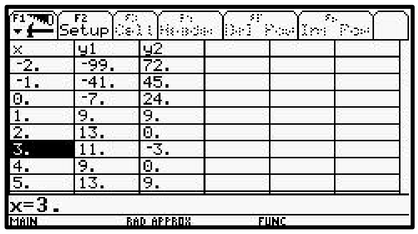
Example. Find all local maxima and minima of the function $g(x)=x^3-9x^2+24x-7$ and sketch graph.





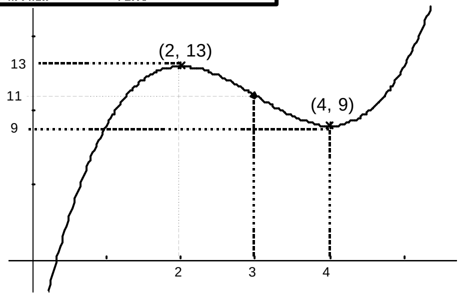






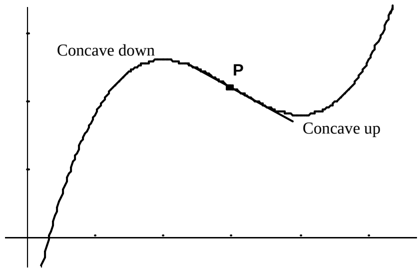
x	y1	y2
-2.	-99.	72.
-1.	-41.	45.
0.	-7.	24.
1.	9.	9.
2.	13.	0.
3.	11.	-3.
4.	9.	0.
5.	13.	9.

x=3.



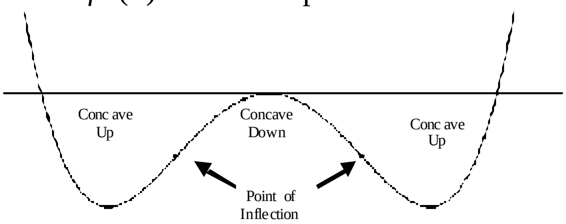
x	$-\infty$	...	2		**3**		4	...	$+\infty$
$g'(x)$		+	0	–	–	–	0	+	
$g(x)$	$-\infty$	↗	**13**	↘	11	↘	**9**	↗	$+\infty$

Concavity
Concave up – if a curve lies *above* its tangent
Concave down – if a curve lies *below* its tangent



Point of **Inflection**
Any point at which the graph of a continuous function changes concavity
Point of Inflection (P) = Concave up + Concave down

- Relationship between $f''(x)$ and point of inflection
  $f''(x) = 0$ in the point of inflection



Exercise. Find the regions of concavity for $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 2$

10. Integration

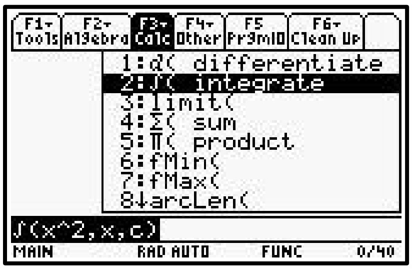
Indefinite integrals

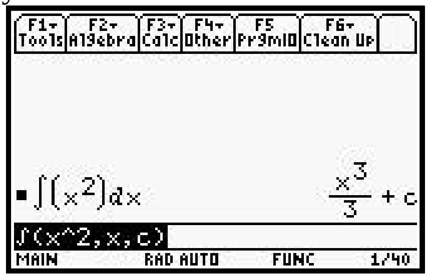
eg) Evaluate $\int x^2 dx$ using the TI–89 by following these steps

[F3] 2 x^2 , x , c) [ENTER]

The [,] [x] tells the calculator to integrate with respect to x

The following should appear on your calculator screen.





Exercise.

Work out the answers to the following.

1. $\int x^3 dx$
2. $\int 2x^3 - 3x^2 + 5dx$
3. $\int (x+3)(x-17)dx$
4. $\int \frac{(2x^2 - 4x)}{2x}dx$

Find an antiderivative for each of the functions:

5. $-\frac{x^2}{\sqrt{2x}}$
6. $-e^{-4x}$
7. $\frac{\tan^2 x - 1}{\sin x}$
8. $\frac{-2e^{-4x} - 1}{3e^{2x}}$

Answers:

1. $\frac{x^4}{4} + c$
2. $\frac{x^4}{2} - x^3 + 5x + c$
3. $\frac{x^3}{3} - 7x^2 - 51x + c$
4. $\frac{x^2}{2} - 2x + c$
5. $\frac{-\sqrt{2}x^{\frac{5}{2}}}{5} + c$
6. $\frac{e^{-4x}}{4} + c$
7. $\frac{\cos x \ln(|\cos(x)+1|) - \cos x \cdot \ln(|\cos(x)-1|) + 2}{2\cos(x)} + c$
8. $\frac{e^{-6x}(3e^{4x} + 2)}{18} + c$

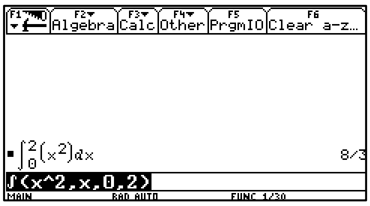
Definite Integrals

Evaluate the definite integral $\int_0^2 x^2 dx$ by following these steps

[F3] 2 x^2 , x , 0 , 2) ENTER

0 = lower limit 2 = upper limit

The following should appear on your screen



Exercise. Work out these definite integrals

1. $\int_0^3 x^3 dx$
2. $\int_{-2}^1 2x^3 - 3x^2 + 5dx$
3. $\int_{-3}^{17} (x+3)(x-17)dx$
4. $\int_1^4 \frac{2x^2 - 4x}{2x} dx$
5. $\int_{\frac{\pi}{6}}^{\frac{3\pi}{2}} \sin(x)dx$
6. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos(x)dx$
7. $\int_{-\frac{1}{2}}^{0} \left(\frac{e^{-2x} - 1}{e^{3x}} \right) dx$

Answers:

1. $\frac{81}{4}$
2. $\frac{-3}{2}$
3. $\frac{-4000}{3}$
4. $\frac{3}{2}$
5. $\frac{\sqrt{3}}{2}$
6. $\frac{-\sqrt{3}}{2} + 1$
7. $\frac{3e^{\frac{5}{2}} - 5e^{\frac{3}{2}} + 2}{15}$

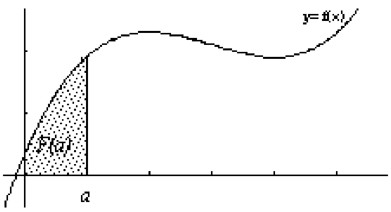
Definite Integrals as Areas

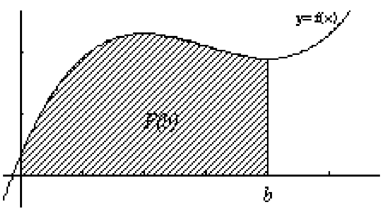
A definite integral written as $\int_a^b f(x)dx$ finds the area between the curve *f(x)* and the *x*-axis, bounded by the lines $x = a$ and $x = b$.

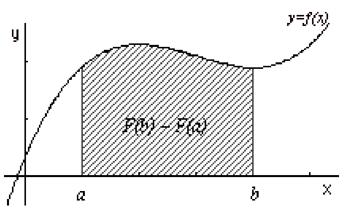
$x = a$ is called the lower limit and $x = b$ is called the upper limit

An alternative method to calculating definite integrals is to graph the function first and then use the $\int f(x)dx$ facility.

We write $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ where $F(x)$ is the antiderivative of $f(x)$







Area from a to $b = F(b) - F(a)$

Total area is $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

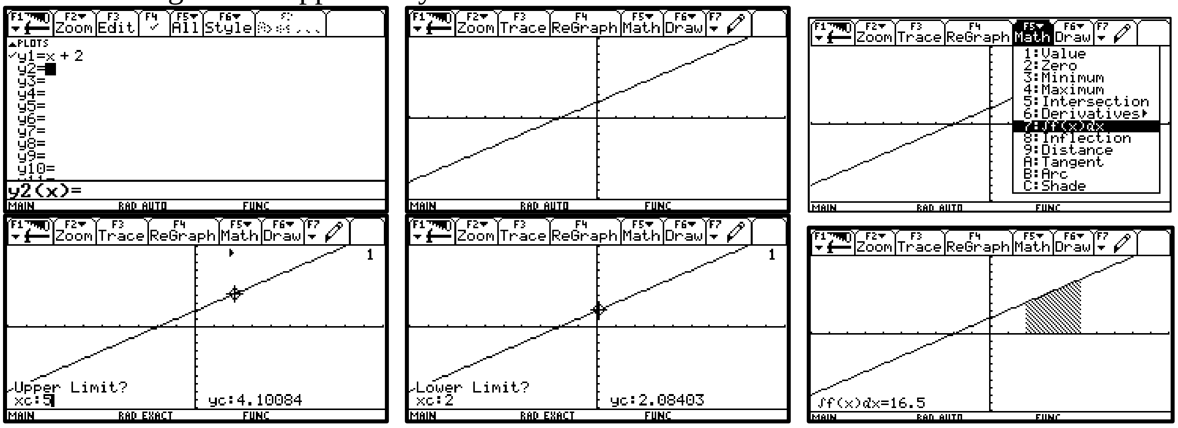
Follow these instructions to find this definite integral $\int_2^5 x + 2dx$.

This method uses the graph of *f(x)* to show the area represented by the integral and numeric integration to calculate it.

◆ [y=] x^2 [ENTER]   [F2] 4   [F5] 7

Note : Only the ***x*** value of the lower and upper limit needs to be typed in. Ignore the *y*-value.

The following should appear on your screen.



Exercise

Follow the above method to represent these integrals as areas between the curve and the *x*–axis and calculate an answer for the definite integral. Use ***y*1=** each time.

1. $\int_{-2}^{2} x - 2dx$
2. $\int_{-3}^{0} x^2 + 3xdx$
3. $\int_{2}^{3} (x+3)(x-2)dx$
4. $\int_{-2}^{2} 4 - x^2dx$
5. $\int_{1}^{3} 4 - x^2dx$

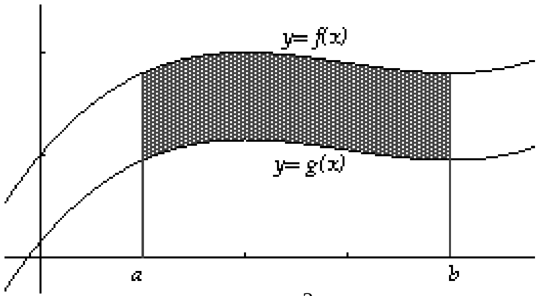
Answers:

1. – 8   2. – 4.5   3. 2.8$\dot{3}$ 4. 10.67

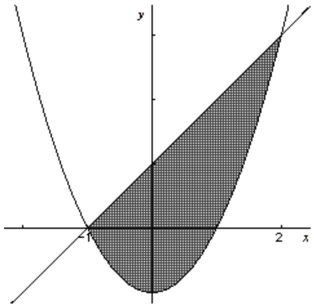
5. – 0.666667

Area between two functions

$$\text{Area} = \int_a^b \{f(x) - g(x)\} dx$$

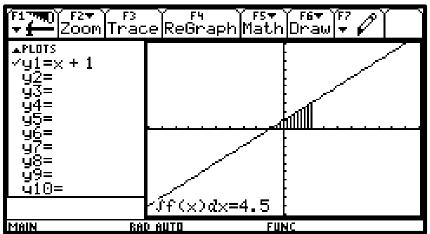


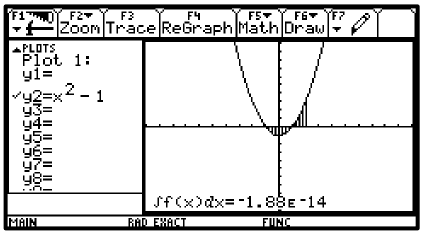
Example 1. Find area between y = x + 1 and y = x^2 – 1

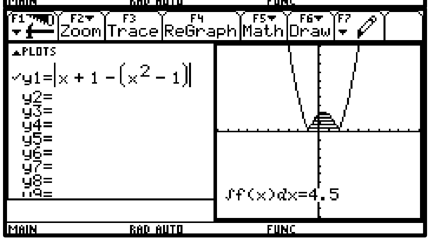


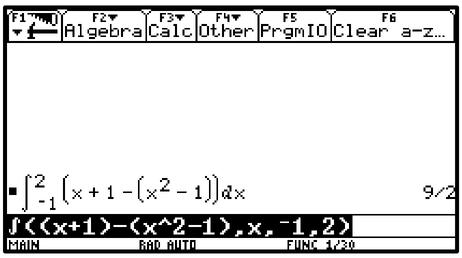
$$S = \int_{-1}^{2} \{(x+1) - (x^2 - 1)\} dx = \int_{-1}^{2} (-x^2 + x + 2) dx = \frac{9}{2} = 4.5$$

For functions with complex intersections we can use $\int_a^b |f(x) - g(x)| \, dx$

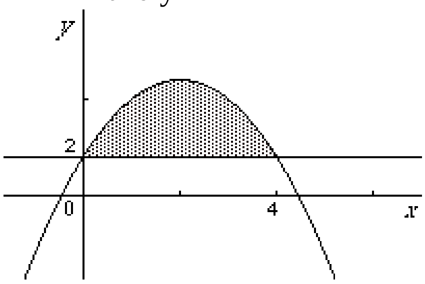




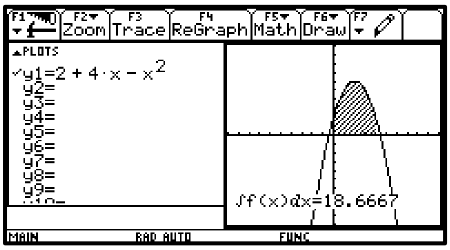


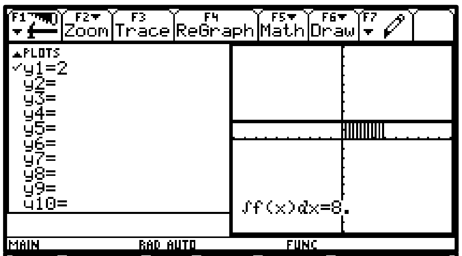


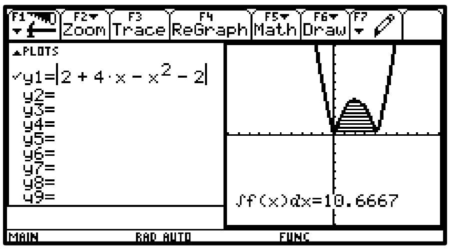
Example 2. Area between y = 2 + 4x – x^2 and y = 2

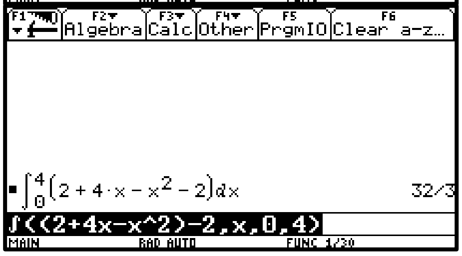


$$S = \int_0^4 \left\{ \left(2 + 4x - x^2\right) - 2 \right\} dx = \int_0^4 (4x - x^2)\,dx = \frac{32}{3}$$



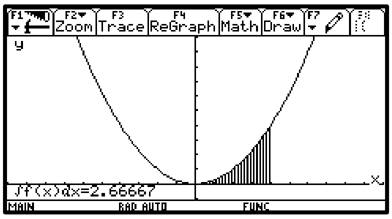


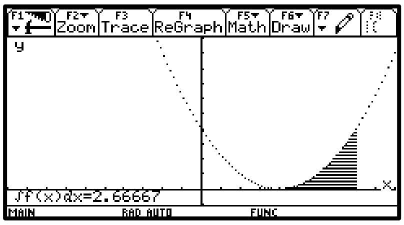


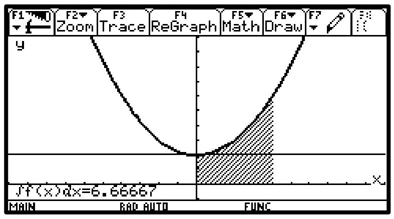


Application

Example. Compare $\int_2^4 (x-2)^2 dx$ and $\int_0^2 (x^2 + 2)dx$ with $\int_0^2 x^2\, dx$







Exercises.

1. Find the area enclosed by the curve $y = 25 - x^2$ and the x-axis. Sketch the graph and shade in this enclosed region. Write down the calculation you need to do to work out the area.

Hint : Use ◆ [F2] to adjust the window range for y-max

2. Find the area enclosed by the curve $y = x^2 - 4x - 5$ and the x-axis. Sketch the graph and shade in this enclosed area. What does the negative sign indicate?

3. Find the area enclosed by the parabola $y = (x-2)^2$, the x–axis and the line x=4.

4. Find the area bounded by the curve $y = x^2 - x + 2$ and the line $y = 8$.

5. The function $f(x) = x(x+1)(x-2)$

   a) Find the area bound by the curve, the x-axis and the lines $x = -1$ and $x = 0$.

   b) Find the area bound by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

   c) Calculate $\int_{-1}^{2} x(x+1)(x-2)dx$.

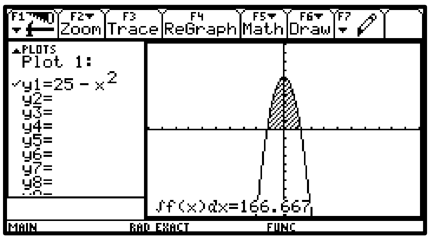
   d) Explain why the answer to c) is not equal to the sum of the 2 areas found in a) and b).

6. a) Using $f(x) = -x^3 + x^2 + 2x$ and $f(x-a)$ for a = 0, 0.5, 1, 1.5, 2 show that $\int_a^{1+a} f(x-a)\,dx$ is constant, and find its value.

   b) Find a formula for $\int_0^k (f(x) + b)\,dx$ and demonstrate it graphically.

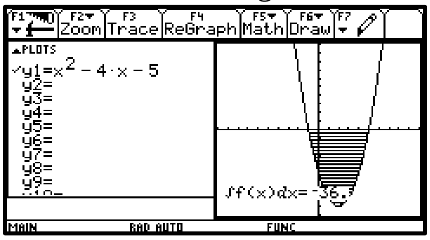
Answers:

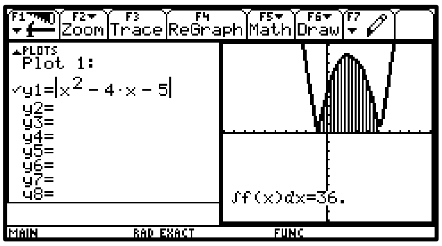
1. $\int_{-5}^{5} 25 - x^2 dx = 166.667$



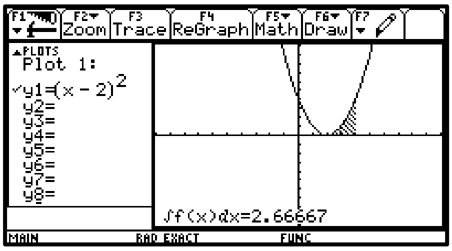
2. $\int_{-1}^{5} \left(x^2 - 4x - 5\right) dx = -36$

Area = 36   Negative indicates the area is below the x-axis

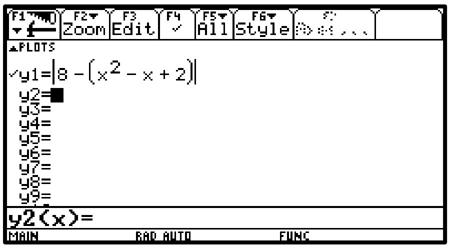


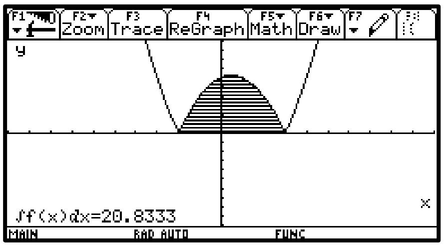


3. $\int_{2}^{4} (x-2)^2 dx = 2.67$



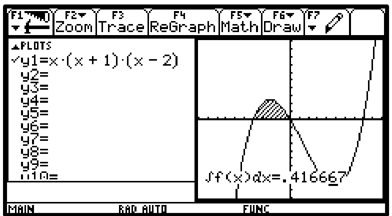
4. Area = $\int_{-2}^{3} \left\{8 - (x^2 - x + 2)\right\} dx = 20.8333$

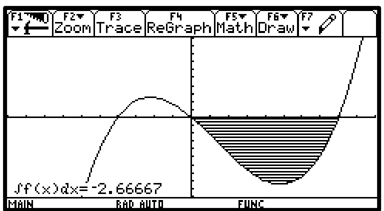


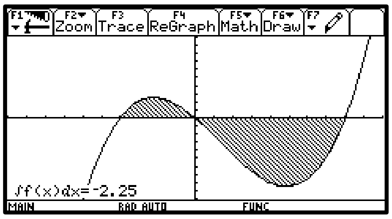


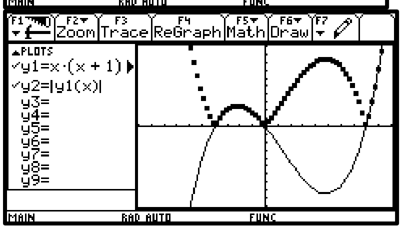
5. a) 0.4167   b) –2.667   c) –2.25

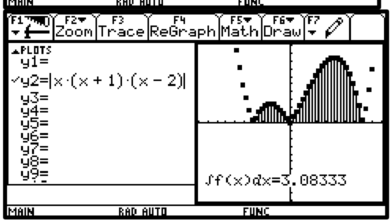
d) Integral does not always equal area. Integrals can be negative. Area is always positive. Area= 0.4167 + |–2.667| =3.0837





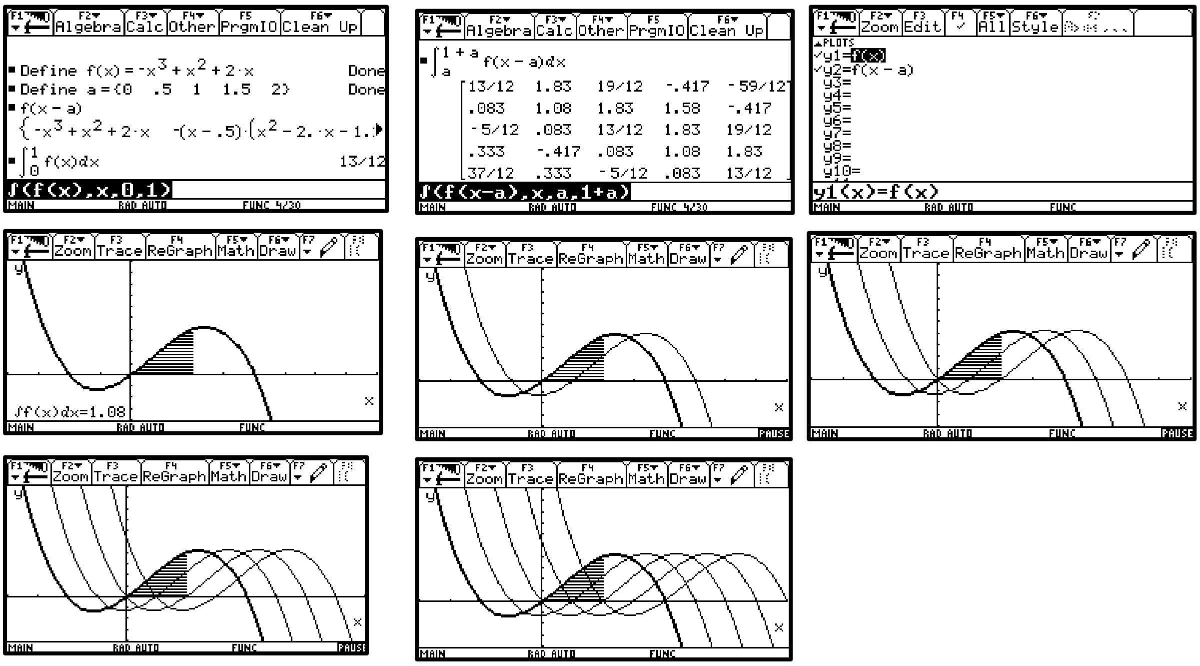






a) Define the function $f(x) = -x^3 + x^2 + 2x$ and $a = \{0, 0.5, 1, 1.5, 2\}$. We can see that $f(x-a)$ is transformed parallel to the x-axis. When we look at the $\int_a^{1+a} f(x-a)\,dx$ along the diagonal of the results, then the values are all the same, that is;

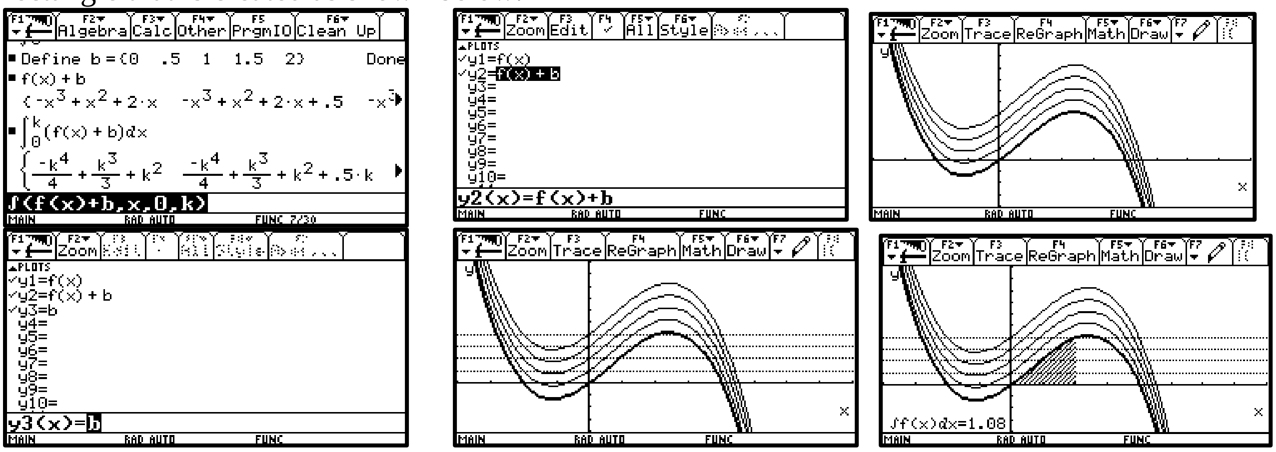
$\int_0^1 f(x-0)dx = \int_{0.5}^{1.5} f(x-0.5)dx = \int_1^2 f(x-1)dx = \int_{1.5}^{2.5} f(x-1.5)dx = \int_2^3 f(x-2)dx$ =13/12=1.08.



b) If we define the value of b = {0, 0.5, 1, 1.5, 2} then *f(x)* + *b* is represented by:

$-x^3 + x^2 + 2x$, $-x^3 + x^2 + 2x + 0.5$, $-x^3 + x^2 + 2x + +1$, $-x^3 + x^2 + 2x + 1.5$, $-x^3 + x^2 + 2x + 2$.

From $\int_0^k (f(x)+b)\,dx$ we can see that the values of 0.5*k*, *k*, 1.5*k*, 2*k* represent area of the extra rectangle that is created as shown below.

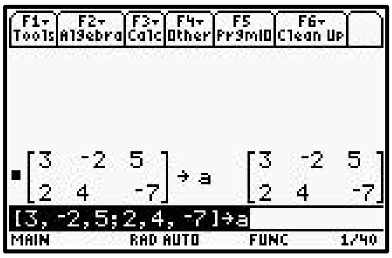


11. Matrices

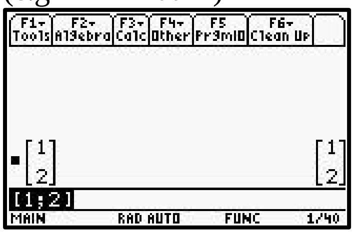
A matrix is simply a convenient way of storing data in an orderly number so that we know the exact position of any piece of data by reference to its row and column.
A matrix is a rectangular array of numbers of the form: A matrix with m rows and n columns is called m ´ n

$$\begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

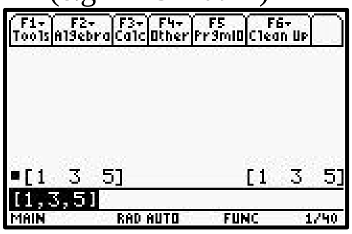
The order of the matrix is determined by the number of rows and columns it contains.
Example. Determine a 2×3 matrix and represent it as A.
[3, –2, 5; 2, 4, -7] STO▸ ALPHA A
Note: The colon (;) separates rows.



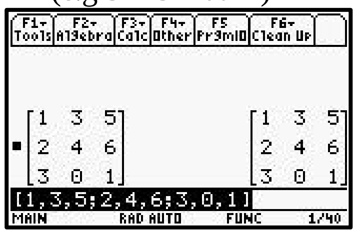
Column matrix
(e.g 2×1 matrix)



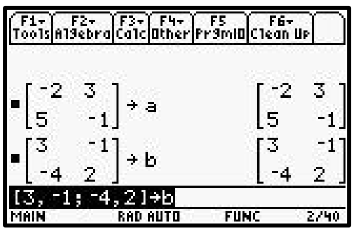
Row matrix
(e.g 1×3 matrix)

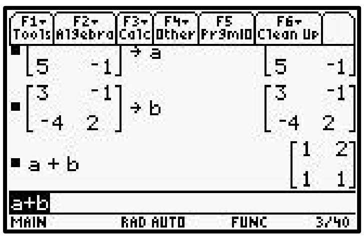


Square matrix
(e.g 3×3 matrix)



Addition of Matrices
Matrices are added by adding elements in corresponding positions.
Matrices can only be added if they are of the same order.
Example. If $A = \begin{bmatrix} -2 & 3 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$ find A + B





$$\begin{bmatrix} -2 & 3 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -2+3 & 3+(-1) \\ 5+(-4) & (-1)+2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Multiplication of Matrices

1. Multiplication by a Scalar

To multiply a matrix by a (scalar) value we multiply every value in the matrix by that value.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Example. Find $-2\begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$.

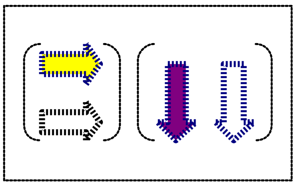
$$-2\begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -6 \\ -8 & -2 & 0 \end{bmatrix}$$

2. Multiplication of a Matrix by a Matrix

- identify the position of the element in the product matrix; e.g. first row, second column
- multiply the elements in the appropriate row in the first matrix by the corresponding elements in the same column of the second matrix.

The product of two 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$



Example. If $A = \begin{bmatrix} 2 & -1 & 4 \\ -1 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & -2 \\ 3 & 2 \end{bmatrix}$ find AB.

Order of A = 2×3 and order of B = 3×2 so order of AB = $2 \times (3 \quad 3) \times 2 \to 2 \times 2$

- identify the position of the element in the product matrix; e.g. first row, second column
- multiply the elements in the appropriate row in the first matrix by the corresponding elements in the same column of the second matrix.

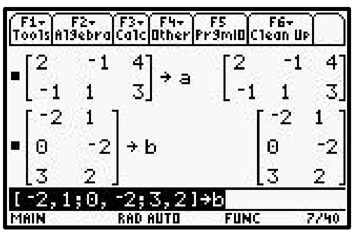
$$AB = \begin{bmatrix} 2 & -1 & 4 \\ -1 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 0 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot(-2)+(-1)\cdot 0+4\cdot 3 & 2\cdot 1+(-1)\cdot(-2)+4\cdot 2 \\ (-1)\cdot(-2)+1\cdot 0+3\cdot 3 & (-1)\cdot 1+1\cdot(-2)+3\cdot 2 \end{bmatrix}$$

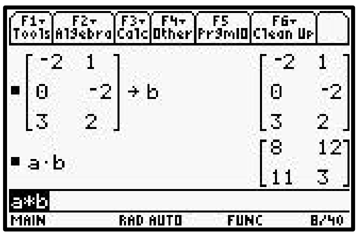
$$= \begin{bmatrix} 8 & 12 \\ 11 & 3 \end{bmatrix}$$

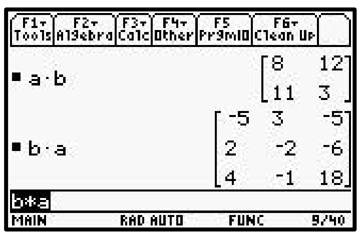
$$BA = \begin{bmatrix} -2 & 1 \\ 0 & -2 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 4 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} (-2)\cdot 2+1\cdot(-1) & (-2)\cdot(-1)+1\cdot 1 & (-2)\cdot 4+1\cdot 3 \\ 0\cdot 2+(-2)\cdot(-1) & 0\cdot(-1)+(-2)\cdot 1 & 0\cdot 4+(-2)\cdot 3 \\ 3\cdot 2+2\cdot(-1) & 3\cdot(-1)+2\cdot 1 & 3\cdot 4+2\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 & -5 \\ 2 & -2 & -6 \\ 4 & -1 & 18 \end{bmatrix}$$

From this we see that *AB ? BA* and so matrix multiplication is not commutative.







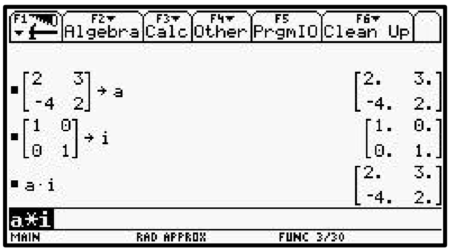
We need the multiplication sign (*) in AB and BA: A*B and B*A

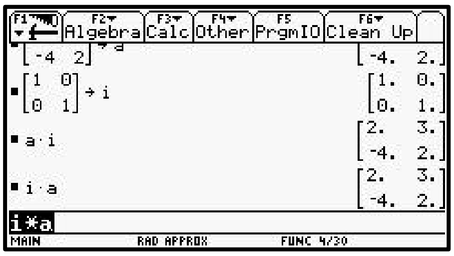
Identity matrix
This is defined as that matrix I for which

$$AI = IA = A$$

This can only happen for $n \times n$ square matrices, with the same number of rows and columns (why?).
Consider

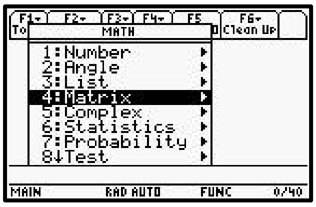
$$\begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$$

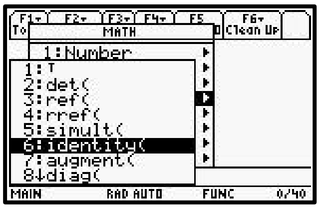




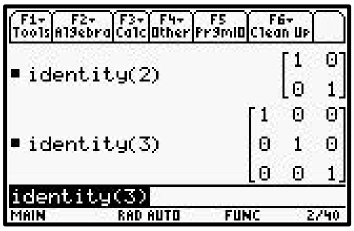
In general the identity matrix is that $n \times n$ matrix with 1s down the diagonal and zeros elsewhere.
On the TI-89 this is obtained by :

[2nd] `5 Option` 4: Matrix `?` 6: identity (2) [ENTER]
then type n) for an $n \times n$ identity. NB n must be a value!



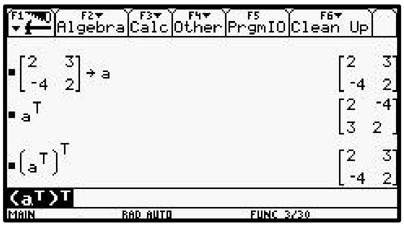


For example:

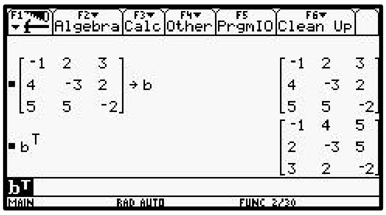


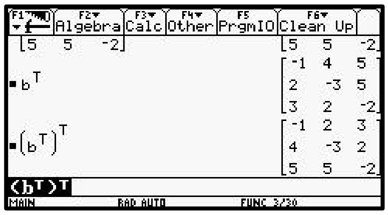
Transpose
The transpose A^T of a matrix A is a matrix formed by interchanging the rows and columns of A.
[2, 3; –4, 2] [STO▸] a [ENTER] a [2nd] `5` Option: 4 Option: 1 (^{T}) [ENTER]

$$\begin{pmatrix} 2 & 3 \\ -4 & 2 \end{pmatrix}^T = \begin{pmatrix} 2 & -4 \\ 3 & 2 \end{pmatrix}$$



$$\begin{pmatrix} -1 & 2 & 3 \\ 4 & -3 & 2 \\ 5 & 5 & -2 \end{pmatrix}^T = \begin{pmatrix} -1 & 4 & 5 \\ 2 & -3 & 5 \\ 3 & 2 & -2 \end{pmatrix}$$



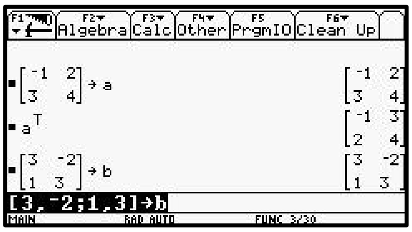


Clearly $(A^T)^T = A$
And also

$$(A + B)^T = A^T + B^T$$

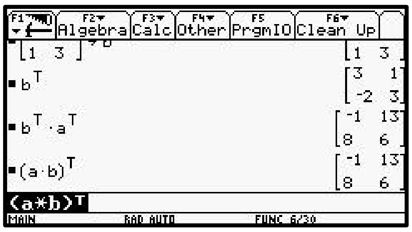
$$(AB)^T = B^T A^T \qquad \text{note order!}$$

Example



$$A=\begin{pmatrix}-1 & 2\\ 3 & 4\end{pmatrix} \text{ then } A^T=\begin{pmatrix}-1 & 2\\ 3 & 4\end{pmatrix}^T=\begin{pmatrix}-1 & 3\\ 2 & 4\end{pmatrix}$$

$$B=\begin{pmatrix}3 & -2\\ 1 & 3\end{pmatrix} \text{ then } B^T=\begin{pmatrix}3 & -2\\ 1 & 3\end{pmatrix}^T=\begin{pmatrix}3 & 1\\ -2 & 3\end{pmatrix}$$



$$B^T A^T=\begin{pmatrix}3 & -2\\ 1 & 3\end{pmatrix}^T\begin{pmatrix}-1 & 2\\ 3 & 4\end{pmatrix}^T=\begin{pmatrix}3 & 1\\ -2 & 3\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & 4\end{pmatrix}=\begin{pmatrix}-1 & 13\\ 8 & 6\end{pmatrix}$$

$$(AB)^T=\left(\begin{pmatrix}-1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}3 & -2\\ 1 & 3\end{pmatrix}\right)^T=\begin{pmatrix}-1 & 8\\ 13 & 6\end{pmatrix}^T=\begin{pmatrix}-1 & 13\\ 8 & 6\end{pmatrix}$$

Determinant of Matrix

For square matrices, of size *n ´ n* we can define a determinant, which will help us find its inverse. The determinant of a 2×2 matrix A:

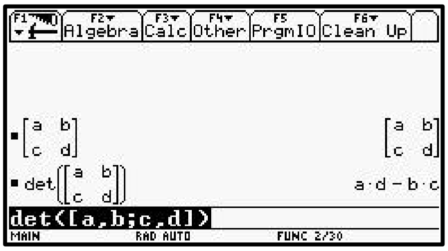
$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

is written

$$\begin{vmatrix}a & B\\ c & D\end{vmatrix}$$

and is given by **det A = *ad* – *bc*.**

On the TI-89 this is obtained by :



[2nd] 5 Option:4 (Matrix)
Option:2 det ([a, b ; c , d])[ENTER]

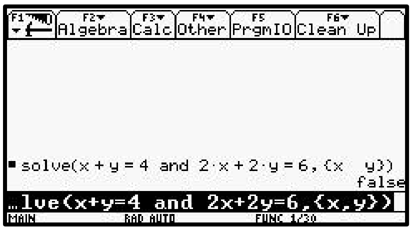
Determinant can be used at the start of a problem on simultaneous equations to check for consistency.

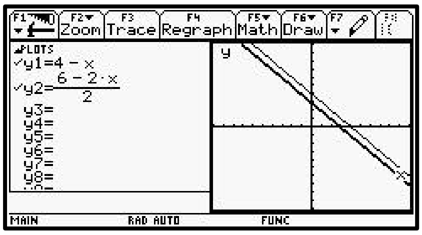
Example. Solve the following sets of simultaneous equations.

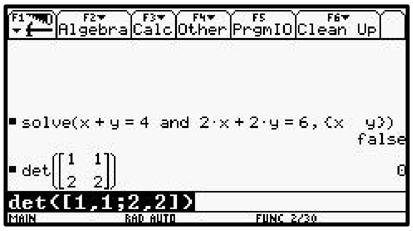
a) $x + y = 4$, $2x + 2y = 6$ b) $2x - y = 3$, $4x - 2y = 6$ c) $4x - 3y = 12$, $x - 2y = -2$

Sol)

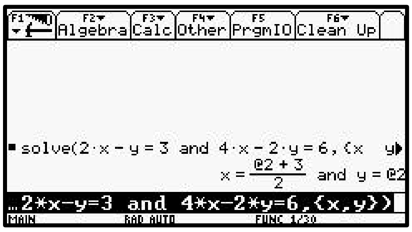
a) No Solution

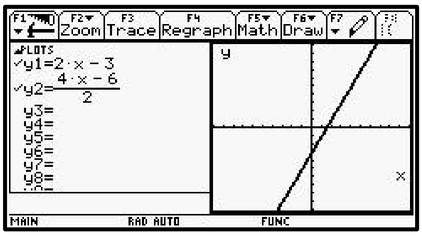


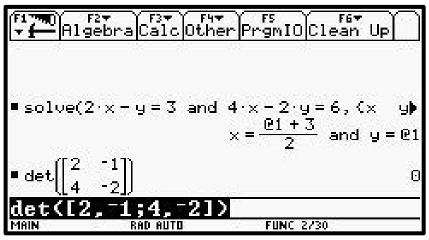




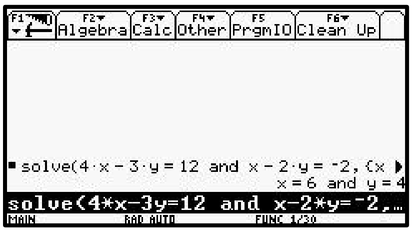
b) Infinitely many solutions

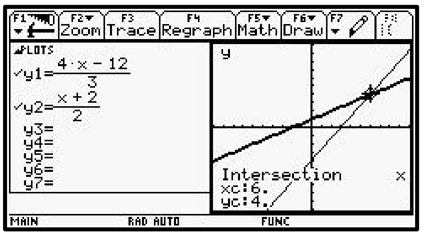


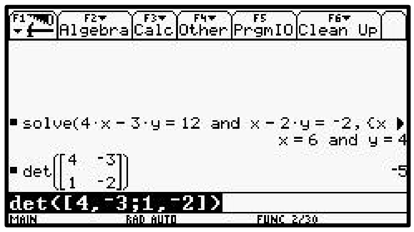




c) Unique solution: $x = 6$ and $y = 4$



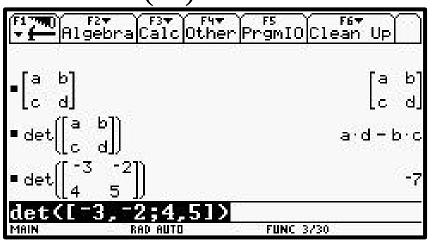




For example

$$\begin{vmatrix} -3 & -2 \\ 4 & 5 \end{vmatrix}$$

$= -15 - (-8) = -15 + 8 = -7$



The determinant of a 3×3 matrix A:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

? Multiply the 3 numbers on each of the leading diagonals (from top left to bottom right): add together from this total, subtract the sum of the products on the other 3 diagonals.

? $\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}, \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$ *and* $\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$ are called **minors**. The minor of an element in a determinant is the determinant formed by omitting the row and column in which the element occurs.

The cofactor of an element is its minor together with its sign. The signs for 3×3 matrix are $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$.

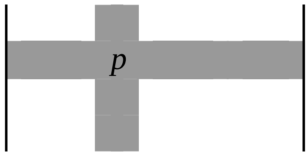
Co–factors

The co–factor of an element in a determinant is the determinant of that the matrix obtained by removing the row and column containing the element from the original determinant, multiplied by +1 or –1 according to the position of the element, as below

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

In general the sign is given by $(-1)^{i+j}$ where the value is in the ith row and the jth column.

For example, remove the shaded cells below to get the cofactor values



Then the value of the determinant for an $n \times n$ matrix is given by

$$det\, A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots a_{1n}C_{1n}$$

For example, given matrix $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12, \quad c_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = 16$$

Adjoint

The matrix of the cofactors of the transpose of a matrix A: *adj A*

Example. Evaluate the determinant, the cofactor and adjoint of matrix A

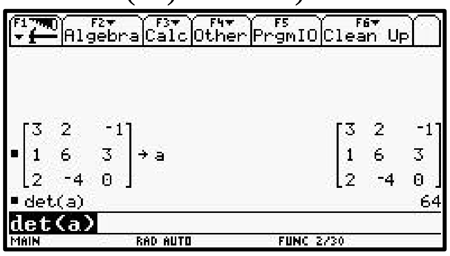
$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

det A = (3×6×0 + 2×3×2 + (–1) ×1× (–4)) – ((–1) ×6×2 +3×3× (–4) + 2×1×0) = 64

On the TI-89 this is obtained by :

[3, 2, –1 ; 1, 6, 3; 2, –4, 0] STO▸ a ENTER

2nd 5 Option:4 (Matrix) Option: 2 det (a) ENTER



Cofactor of matrix A = $\begin{bmatrix} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -2 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$

adj A = $\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$

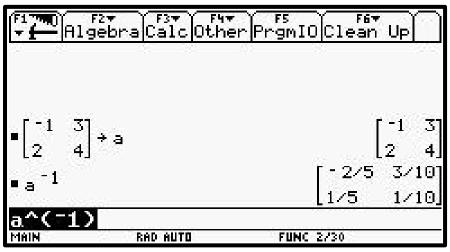
Inverses

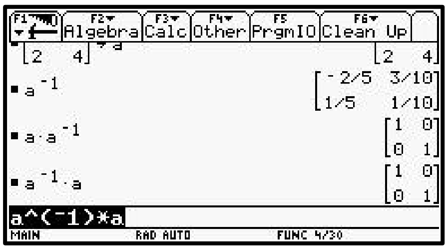
We define the inverse of a matrix A to be that matrix A^{-1} such that:

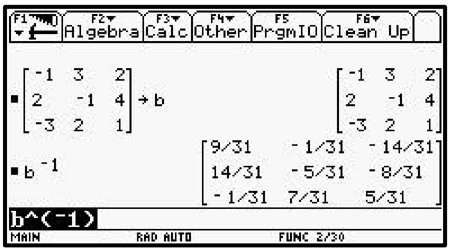
$$A\,A^{-1} = A^{-1}A = I_n$$

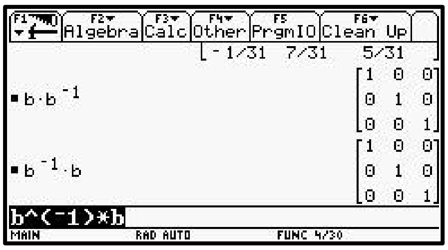
Where I_n is the $n \times n$ identity matrix.

Thus only square matrices can have inverses.



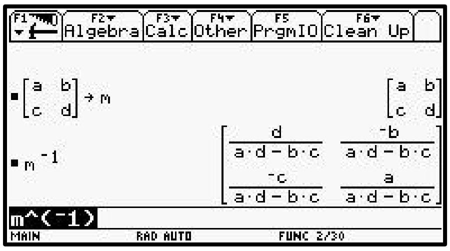


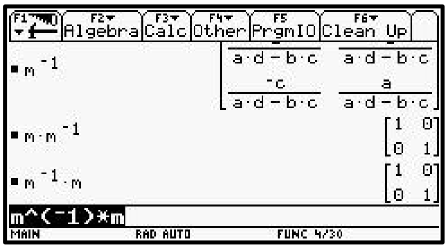




Inverses for 2´2 matrices

For 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$





To solve the simultaneous equations as a single matrix equation:

Write the system of equation as a single matrix equation

$ax + by = c$

$cx + dy = f$ becomes $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

If we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

And A^{-1} be the inverse of A, then we can multiply both sides of equation 1 by A^{-1}, giving

$$A^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$$

But $A\,A^{-1} = I$ by definition

So $I\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$

And hence the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$,

where $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

To solve simultaneous equations we multiply the number matrix by the inverse of the coefficient matrix, if it exists (if det A? 0).

If the matrix of coefficient is singular (the determinant $ad – dc = 0$), the simultaneous equations represent either two parallel lines or two lines which are coincident.

Example: Solve the simultaneous equations $4x – 3y = 12$ and $x – 2y = –2$.

In matrix form: $\begin{bmatrix}4 & -3\\1 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}12\\-2\end{bmatrix}$

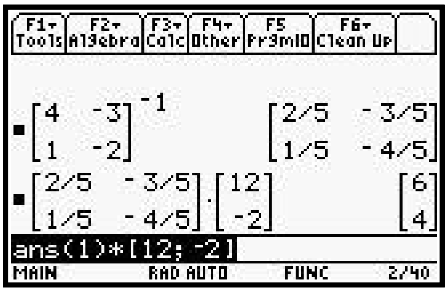
$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4 & -3\\1 & -2\end{bmatrix}^{-1}\begin{bmatrix}12\\-2\end{bmatrix}$$

$$=\frac{1}{4\cdot(-2)-(-3)\cdot 1}\begin{bmatrix}-2 & 3\\-1 & 4\end{bmatrix}\begin{bmatrix}12\\-2\end{bmatrix}$$

$$=-\frac{1}{5}\begin{bmatrix}-2\cdot 12+3\cdot(-2)\\(-1)\cdot 12+4\cdot(-2)\end{bmatrix}$$

$$=-\frac{1}{5}\begin{bmatrix}-30\\-20\end{bmatrix}$$

$$=\begin{bmatrix}6\\4\end{bmatrix}$$



So $x = 6$, $y = 4$

Inverses for 3´3 matrices

Use matrix methods to solve

$$3x – y + 2z = 13$$
$$–x + 4y + 2z = –1$$
$$4y + 3z = 4$$

Writing in matrix form: $\begin{bmatrix}3 & -1 & 2\\-1 & 4 & 2\\0 & 4 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}13\\-1\\4\end{bmatrix}$

To find the inverse of a 3×3 matrix we carry out the following steps.

Given a matrix A, for example, $A=\begin{bmatrix}3 & -1 & 2\\-1 & 4 & 2\\0 & 4 & 3\end{bmatrix}$

Step1. Define A^T, the transpose of A

$$A^T=\begin{bmatrix}3 & -1 & 0\\-1 & 4 & 4\\2 & 2 & 3\end{bmatrix}$$

Step 2. Obtain the *adjoint* matrix, written *adj A*, by replacing each element in the transpose of A by its cofactor, and by changing the sign of every second element.

$$\text{adj } A=\begin{bmatrix}\begin{vmatrix}4 & 4\\2 & 3\end{vmatrix} & -\begin{vmatrix}-1 & 4\\2 & 3\end{vmatrix} & \begin{vmatrix}-1 & 4\\2 & 2\end{vmatrix}\\ -\begin{vmatrix}-1 & 0\\2 & 3\end{vmatrix} & \begin{vmatrix}3 & 0\\2 & 3\end{vmatrix} & -\begin{vmatrix}3 & -1\\2 & 2\end{vmatrix}\\ \begin{vmatrix}-1 & 0\\4 & 4\end{vmatrix} & -\begin{vmatrix}3 & 0\\-1 & 4\end{vmatrix} & \begin{vmatrix}3 & -1\\-1 & 4\end{vmatrix}\end{bmatrix}=\begin{bmatrix}4 & 11 & -10\\3 & 9 & -8\\-4 & -12 & 11\end{bmatrix}$$

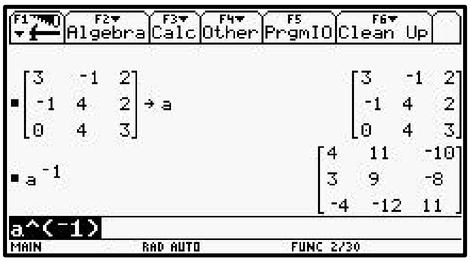
Step 3. Find *det A*

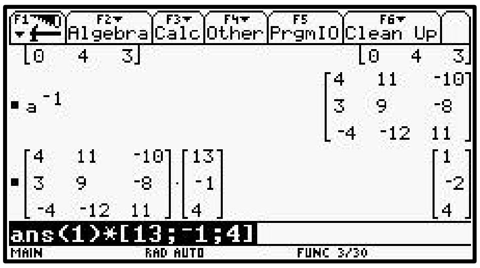
$\det A = (3×4×3 + (–1)×2×0 + 2×(–1)×4) – (2×4×0 + 2×4×3 + 3×(–1)×(–1)) = 36 – 8 – (24 + 3) = 1$

$$A^{-1} = \frac{1}{\det A} \cdot adj(A) = \frac{1}{1}\begin{bmatrix} 4 & 11 & -10 \\ 3 & 9 & -8 \\ -4 & -12 & 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 11 & -10 \\ 3 & 9 & -8 \\ -4 & -12 & 11 \end{bmatrix}\begin{bmatrix} 13 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Thus $x = 1$, $y = -2$ and $z = 4$





<u>Systems of linear equations</u>

A solution to a system of linear equations gives the corresponding values of each of the variables that satisfy all the equations simultaneously.
Solving Systems of Equations
There are three things we can do to a system of equations which do not alter their solutions:

1. Interchange any two equations
2. Multiply any equation through by a constant (≠0)
3. Add a constant multiple (≠0) of any equation to any other equation.

Gaussian Elimination

When there are 3 equations – in x, y, and z – we start by eliminating the first variable (x) in the last 2 equations and then eliminate the second variable (y) in the last equation. This leaves us with a set of equations in *upper triangular form*, or ***echelon form***.
Once the equations are *echelon form*, they can be solve by **back substitution**.
The leading variable in each equation in the list falls further to the right each time.
The rules become:

1. Interchange any two rows
2. Multiply any row through by a constant (≠0)
3. Add a constant multiple (≠0) of any row to any other row.

Example.

$3x - y + 2z = 13$…………R1
$-x + 4y + 2z = -1$……….R2
$4y + 3z = 4$…………….R3

The augmented matrix is: $\begin{bmatrix} 3 & -1 & 2 & 13 \\ -1 & 4 & 2 & -1 \\ 0 & 4 & 3 & 4 \end{bmatrix}$

To make echelonform, [3, –2, 2, 13 ; –1, 4, 2, –1 ; 0, 4, 3, 4] STO▸ a ENTER
R1 + 3R2

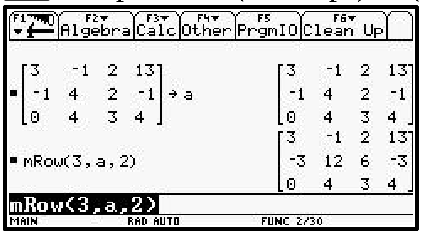
$$\begin{bmatrix} 3 & -1 & 2 & 13 \\ 0 & 11 & 8 & 10 \\ 0 & 4 & 3 & 4 \end{bmatrix}$$

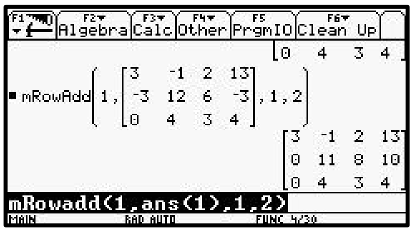
–4R2 + 11R3

$$\begin{bmatrix} 3 & -1 & 2 & 13 \\ 0 & 11 & 8 & 10 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

[2nd] 5 Option: 4 (Matrix) J (Row ops) 3(mRow) (3, a, 2) [ENTER]

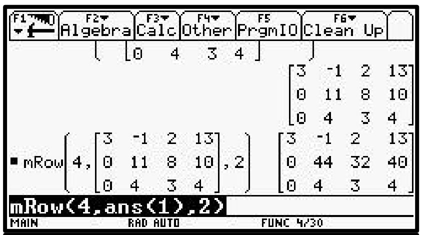
[2nd] 5 Option:4 J (Row ops) 4 (mRowAdd (1, ans(1), 1, 2) [ENTER]

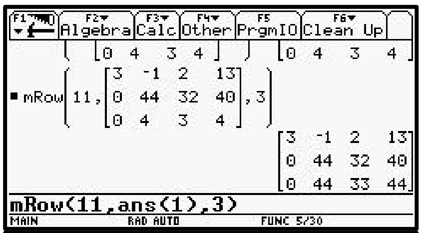


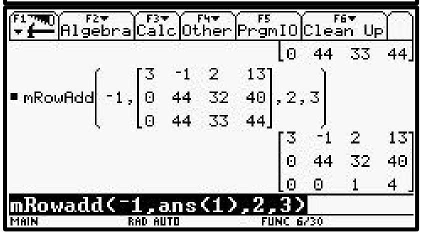


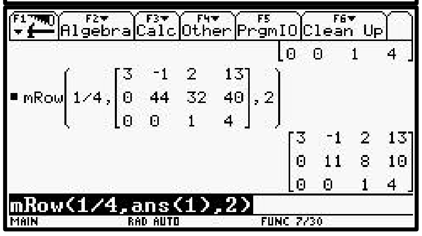
$$3x - y + 2z = 13$$
$$11y + 8z = -1$$
$$z = 4$$

By back substitution,
$z = 4$,
$11y + 8(4) = -1$, so $y = -2$
$3x - (-2) + 2(4) = 13$, so $x = 1$









$$\begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & \frac{3}{4} & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

On the TI-89 this is obtained by:
[3, –2, 2, 13 ; –1, 4, 2, –1 ; 0, 4, 3, 4] [STO▸] a [ENTER]

[2nd] 5 Option 4 (Matrix) 3(ref) (a) [ENTER]