Ages 15-17 – Heron's formula

The following tested teaching sequence will show you:

- *how students are able to prove Heron's formula, a rather hard task for most of the students without CAS-calculator,*
- *how Heron's formula can be used as a functional module (functional thinking),*
- *how the usual pure calculation exercises about Heron's formula can be replaced by nice and demanding ones, which are leading to reflection and a deeper understanding of mathematics,*
- *how a new dynamic is given to tasks by parametrisation.*

1. Worksheet: Heron's Formula

A triangle can be constructed with three sides given; therefore its area is also determined. The Greek mathematician Heron of Alexandria found the following formula for calculating the area of a triangle from its sides:

Heron's formula for the area of a triangle

$$area = \sqrt{s(s-a)(s-b)(s-c)} \text{ with } s = \frac{a+b+c}{2} \text{ (half of the circumference)}$$

Exercise 1 (Category C1)

Store the above formula as a function named `heron(a,b,c)`.
Attach the additional condition with `|`.

Calculate the area of the right triangle with sides 3, 4, 5.

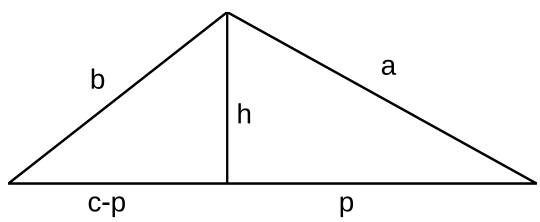
Calculate the area of some other triangles (choose the length of the sides yourself).
Interpret possible error messages.

Calculate the side c of a triangle from $area = 5$, $a = 3$ and $b = 4$.

Storing the formula as a function and applying it like a blackbox is a very efficient method when using CAS.

Exercise 2

Prove Heron's formula using the figure beside and the following guidelines
(insert your results in the worksheet)!



Proof of Heron's formula

An elementary algebraic proof of Heron's formula is rather sophisticated and is difficult for students to do without help. The following proof was done individually with CAS and the following worksheet.

Follow the instructions.

a) We calculate the square of Heron's formula $area^2$ = `heron(a,b,c)`^{2} in order to avoid roots.

The CAS-calculator returns (s has been replaced by $\frac{a+b+c}{2}$):

$$area^2 = \frac{-(a+b+c)\cdot(a+b-c)\cdot(a-b-c)\cdot(a-b+c)}{16}$$

which has to be proven.

b) For the area of the triangle, we start with a well-known formula using baseline c and height h:

$$area = \frac{c \cdot h}{2} \quad \text{or} \quad area^2 = \frac{c^2 \cdot h^2}{4}$$

c) c is given. Express h^2 in terms of a and p and then in terms of b and c-p using the Pythagorean theorem:

h^2 = (using a and p, store this in `hsqr`)
h^2 = (using b and c-p)

d) Equate both expressions for h^2 and solve with respect to p.

Check your answer: $p = \frac{a^2 - b^2 + c^2}{2c}$

e) Factor `hsqr` from c) and substitute for p from d) (CAS: `factor()|p=..`).

Store the result in `hsqr` (= h^2) and insert it in $area^2 = \frac{c^2 \cdot \text{hsqr}}{4}$ from b).

Compare your answer with the expression obtained in a).

2. Further Exercises

Exercise 3 (Category C3)

Calculate the area of a trapezoid with sides a = 6, b = 4, c = 3 and d =3.
Define a general function `trapez(a,b,c,d)`.

Exercise 4 (Category C4)

Does there exist a triangle with sides x, x + 1, x + 2 and area T = 10?
For which given area T does a solution exist? (An example from my colleague H.R. Schneider)

Exercise 5 (Category C4)

Does a triangle exist with sides 3, x, 2x and area T = 10?

Does a triangle with the same sides exist for T = 1, 3, 0?

For which given area T does a solution exist?

Store the function `heron(3,x,2x)` [STO▸] `y1(x)` , draw the function and determine an approximate domain and range.

Draw the vertices C of possible triangles ABC with $|AB| = 3$. On which curve do these vertices lie?

You can also construct the locus using a dynamic geometry system.

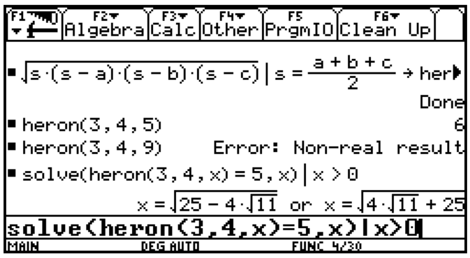
Exercise 6 (Category C0)

Who was Heron of Alexandria and what were his most important achievements? Where is Alexandria? What was Alexandria renowned for? Search for information to answer these questions in a dictionary, on the Internet or using other sources.

3. Solutions to the exercises

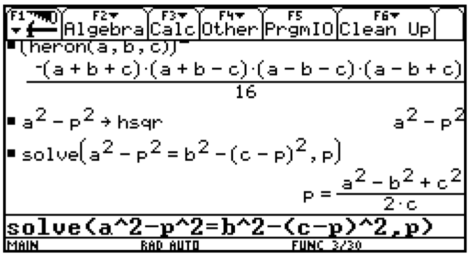
Exercise 1

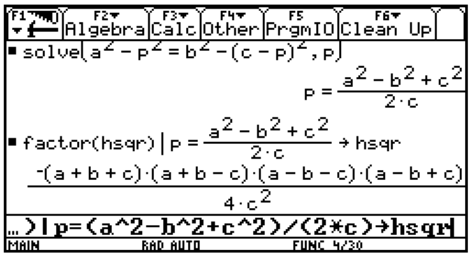
If one side is larger than the sum of the others, a real solution doesn't exist. The latter example shows you that there exists at most 4 solutions to the equation, where two of them correspond to a triangle ($x > 0$). It is worth drawing the triangles, determining the height by calculation.

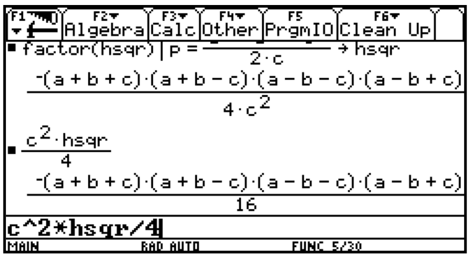


Exercise 2

You can see the steps of the proof on the screenshots below.

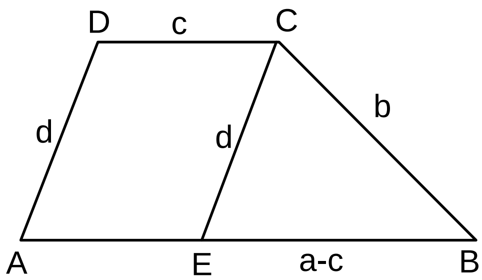






The proof with CAS is much easier than by hand, because the most difficult part is to transform the last expression to the form of Heron's formula. CAS simplifies Heron's formula automatically tothe last expression. By the way, CAS shows you,how to perform the proof in an easier way by-hand as well.

Exercise 3



We calculate the height of the trapezoid using the triangle *EBC*.

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solve(heron(abs(a-c),b,d)
=h*abs(a-c)/2,h)
```

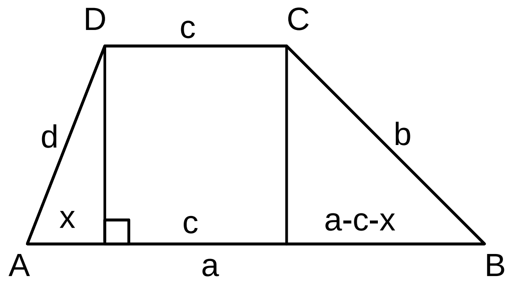
Then we calculate the area by $\frac{(a+c)\cdot h}{2}$.

This general formula is very extensive and can be stored as a function `trapez(a,b,c,d)`, e.g. `trapez(3,3,6,4)=6√5`. Why does the function fail for rectangles!

A student proposed another solution without using absolute values (similar to the proof of Heron's formula): $d^2 - x^2 = b^2 - (a - c - x)^2 \ (= h^2)$.

Solve this equation for x and substitute x into the formula for the trapezoid:

$$\frac{(a+c)\cdot h}{2} = \frac{(a+c)\cdot\sqrt{d^2 - x^2}}{2}$$



Exercise 4

`solve(heron(x,x+1,x+2)=10,x)` has the solution x = 4.018 (and a negative one).

The CAS is overtaxed with `solve(heron(x,x+1,x+2)=t,x)` (equation of order 4).

With geometrical considerations you can see that for large values of x a triangle always exists and is nearly equilateral. At the other end you get a triangle with area 0 when $x = 1$. For $x > 1$ a triangle always exists with continuously increasing area.

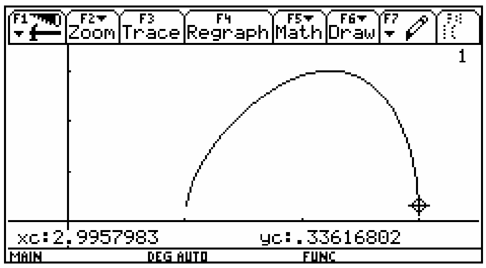
You can also show the numerical effects of rounding, e.g. with c = 0.00001, the result is false.

Exercise 5

`solve(heron(3,x,2x)=10,x)`
Result: `false`

For T = 1, 3, 0 you get 1.109 or 2.962, √5, 1 or 3 (positive values only).

Using trace or the table function you can determine the domain [1,3] and the range [0,3].



The vertices C of possible triangles with A = (0;0) and B = (3;0) are lying on the Apollonian circle over the segment $\overline{AB}$ with ratio 1:2. This is the circle of Thales over the segment with endpoints (2;0) and (6;0).

Note: This exercise can be done at an elementary level (9th school year). I did it after having intro duced the Apollonian circle as an application of the intercept theorems. It can also be used as an experimental introduction to the Apollonian circle.

Exercise 6

Heron of Alexandria was a Greek scientist. We are not sure about his exact date of birth but we do know that it is between 150 B.C. and 250 A.C. Most important works: Mechanika, Pneumatika. Alexandria lies near the delta of the Nile in Egypt and is considered the most important scientific centre of the Hellenistic age. The Alexandria library was very famous with nearly one million book rolls.

4. Exam questions

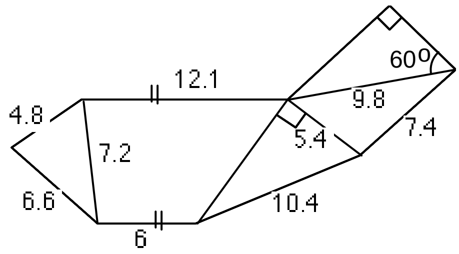
a) Category C1

Use Heron's formula to find the area T of the triangle with vertices A = (13; 2), B = (5; 17) and C = (22; -4).

b) Category C1
Use Heron’s formula to calculate c and the area T of a triangle with $a = 7$, $b = 5$ and $h_c = 3$.

c) Category C1
In a consolidation of property a farmer can exchange his agricultural land (see figure beside, measures in km) for a square piece of land.

What is the length of the sides of this land?



d) Category C1
Does a triangle exist with sides 1, 2, x and area 10?

Draw `heron(1,2,x)` STO▶ `y1(x)`.

Use ‘Trace’ to find an approximate domain of $y1(x)$ ($x>0$). State the reason why x has to lie in this domain (geometrical consideration).

What is the largest possible area? Give a reason for your answer!

e) Category C1
Let us consider triangles with sides 4, x, and $k \cdot x$ ($k \geq 1$).

For $k = 3$, determine the locus of the vertices of these triangles (baseline 4) and the largest possible area. (Tip: First consider triangles with area 0.)

What is the locus of the vertices for an arbitrary value for k?

What is the locus for $k = 1$?

5. Solutions to the exam questions

a) Store the vertices of the triangle: `[-13,2]` STO▶ `a,[5,17]` STO▶ `b,[22,-4]` STO▶ `c`.
`heron(norm(b-a),norm(c-b),norm(c-a))`, result: 316.5.

b) First calculate side c: `zeros(heron(7,5,c)-3*c/2,c)`, result (numeric) {2.325,10.325}. We then get T with: `3*c/2|c={2.325,10.325}`, result {3.487,15.487}.

Note: The command `zeros` returns a list, which allows you to calculate several values for a variable simultaneously.

c) If the modules `heron(a,b,c)` and `trapez(a,b,c,d)` are not still stored in the CAS, get students to define them again.

First calculate the missing side, s, of the trapezoid: `10.4^2-5.4^2` STO▶ `s`.

The rest is an application of the functions heron and trapez:

`√(heron(4.8,7.2,6.6) + trapez(12.1,7.2,6,s) + 5.4*s/2 + heron(5.4,7.4,9.8)`
`+ 4.9^2*√(3)/2)`, result: 12.028 km.

d) `solve(heron(1,2,x)=10,x)`. The result given is `false`.

Domain [1,3] (read from the graph of `heron(1,2,x)` using the trace feature).

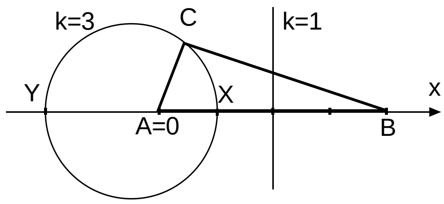
Choose $\overline{AB}$ as the side with length 1 of the triangle and $\overline{AC}$ with length 2. Then the vertices C are on a circle with radius 2 and centre A.

For all circles with radius x and centre B the area will be 0 if the circles are touching. This is for the cases x = 1 (smallest possible value) and x = 3 (largest possible value).

The triangle has the largest height and hence the largest area if side $\overline{AC}$ is perpendicular to $\overline{AB}$, i.e. A = 1. To illustrate the situation dynamically you can do the construction with Cabri.

e) The locus is the Apollonian circle over the side with length 4 and ratio 1: k. You find the diameter of the circle at area 0 by entering `solve(heron(4,x,k*x)=Ø,x)`and this gives the result

$x = \pm\frac{4}{k+1}$ and $x = \pm\frac{4}{k-1}$, where $\frac{4}{k+1}$ corresponds to the inner intersection point X and $-\frac{4}{k-1}$ to the outer intersection point Y. For discussion use an x-axis through A and B with A = 0.



For k = 3 the x-values are 1 and -2, i.e. the Apollonian circle has radius 1.5 and the largest area is therefore 3.

For an arbitrary k the radius is half of the difference of the above values: $\frac{4k}{(k-1)\cdot(k+1)}$.

For $k \to \infty$ (or k = 0) the Apollonian circle shrinks to point A (or B). For k = 1 you get an isosceles triangle with sides 4, x and x.

These ideas will challenge and expand students' minds as they consider the geometry associated with the limit $k \to 1$ of the Apollonian circle with radius and centre point tending to "infinity". The extension to the case $0 \le k \le 1$ may be difficult for students to understand geometrically. However, it is useful to extend students' thinking by getting them to consider what happens near k=1 with the circle and its centre by "passing through infinity" from $-\infty$ to $+\infty$.