Mathematical Methods (CAS) 2003 Examination 2 sample solutions

Question 3

The function f should first be defined.

#1: $f(x) := x^3 \cdot e^{-2 \cdot x}$

a. Find f’(x)

#2: $f'(x)$

#3: $$x^2 \cdot e^{-2 \cdot x} \cdot (3 - 2 \cdot x)$$

b. Solve f’(x)=0 to find te x–coordinates of the stationary points. Substitute these values to get the y–coordinates.

#4: $f'(x) = 0$

#5: $$x^2 \cdot e^{-2 \cdot x} \cdot (3 - 2 \cdot x) = 0$$

#6: $\text{SOLVE}(x^2 \cdot e^{-2 \cdot x} \cdot (3 - 2 \cdot x) = 0, x, \text{Real})$

#7: $$x = \infty \vee x = \frac{3}{2} \vee x = 0$$

#8: $f(0)$

#9: $$0$$

#10: $f\left(\frac{3}{2}\right)$

#11: $$\frac{27 \cdot e^{-3}}{8}$$

The stationary points are
(0,0) a stationary point of inflection
(3/2,27e^(–3)/8) a local maximum.

c. i. Find f(1) and f’(1) and write down the equation of the tangent, or, alternatively, use the TANGENT function.

#12: f(1)

#13: e^{-2}

#14: f′(1)

#15: e^{-2}

Hence equation of tangent is y−e^(−2)=e^(−2)(x−1) which simplifies to y=xe^(−2). Alternatively,

#16: y = TANGENT(f(x), x, 1)

#17: $y = x\cdot e^{-2}$

c. ii. It is fairly easy to see that f(0)=0 and f′(0)=0, so equation of tangent will be y=0. This supports previous information (such as graph, (0,0) being a stationary point of inflection). If we use the TANGENT function:

#18: y = TANGENT(f(x), x, 0)

#19: y = 0

Any non-vertical line through the origin has equation y = mx, where m is the slope. If this line is a tangent to the curve at (x,y), then m = f(x)/x=f′(x). Solve this equation for x.

#20: $\frac{f(x)}{x} = f'(x)$

#21: $x^2\cdot e^{-2\cdot x} = x^2\cdot e^{-2\cdot x}\cdot(3 - 2\cdot x)$

#22: $\frac{f(x)}{x} - f'(x) = 0$

#23: $2\cdot x^2\cdot e^{-2\cdot x}\cdot(x - 1) = 0$

In the above form, I can see that the only solutions are x=0 and x=1. Solve it anyhow:

#24: SOLVE($2\cdot x^2\cdot e^{-2\cdot x}\cdot(x - 1) = 0$, x, Real)

#25: $x = \infty \lor x = 1 \lor x = 0$

Hence there are only two tangents that pass through the origin, and they are the tangents at x=0 and x=1.

d. i. If g is a porbability density function, then the integral of g from 0 to infinity must be 1. Use this to solve for k.

#26: $k\cdot f(x)$

#27: $$k\cdot x^{3}\cdot e^{-2\cdot x}$$

#28: $\int_{0}^{\infty} k\cdot x^{3}\cdot e^{-2\cdot x}\, dx$

#29: $$\frac{3\cdot k}{8}$$

#30: $\frac{3\cdot k}{8} = 1$

#31: $$\frac{3\cdot k}{8} = 1$$

#32: $\text{SOLVE}\left(\frac{3\cdot k}{8} = 1, k, \text{Real}\right)$

#33: $$k = \frac{8}{3}$$

d. ii. If m is the median, then the integral of g from 0 to m equals 0.5. Solve this for m.

#34: $\frac{8}{3}\cdot f(x)$

#35: $$\frac{8\cdot x^{3}\cdot e^{-2\cdot x}}{3}$$

#36: $$\int_0^m \frac{8\cdot x^3\cdot e^{-2\cdot x}}{3}\,dx$$

#37: $$1-\frac{e^{-2\cdot m}\cdot(4\cdot m^3+6\cdot m^2+6\cdot m+3)}{3}$$

#38: $$1-\frac{e^{-2\cdot m}\cdot(4\cdot m^3+6\cdot m^2+6\cdot m+3)}{3}=0.5$$

#39: $$1-\frac{e^{-2\cdot m}\cdot(4\cdot m^3+6\cdot m^2+6\cdot m+3)}{3}=\frac{1}{2}$$

If I solve (numerically) this directly, Derive gives me a negative answer. I need to specify a lower bound of 0.

#40: $$\text{NSOLVE}\left(1-\frac{e^{-2\cdot m}\cdot(4\cdot m^3+6\cdot m^2+6\cdot m+3)}{3}=\frac{1}{2}, m, 0, 10\right)$$

#41: $$m = 1.836030334$$

Answer is 1.84, correct to 2 decimal places.

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Question 4

First define the function

#42: $$x(t) :=\frac{3\cdot t}{5+t^2}$$

a. Concentration will be greatest when x'(t)=0.

#43: $$x'(t) = 0$$

#44: $$\frac{3\cdot(5-t^2)}{(t^2+5)^2}=0$$

#45: $$\text{SOLVE}\left(\frac{3\cdot(5-t^2)}{(t^2+5)^2}=0,\ t\right)$$

#46: $$t=\pm\infty \lor t=-\sqrt{5} \lor t=\sqrt{5}$$

#47: $x(\sqrt{5})$

#48: $$\frac{3\cdot\sqrt{5}}{10}$$

Greatest concentration is $3\sqrt{5}/10$, when $t=\sqrt{5}$.
b. Find x'(1)

#49: $x'(1)$

#50: $$\frac{1}{3}$$

c. Find the times when the concentration is 0.4, and subtract them.

#51: $x(t) = 0.4$

#52: $$\frac{3\cdot t}{t^2+5}=\frac{2}{5}$$

#53: $$\text{SOLVE}\left(\frac{3\cdot t}{t^2+5}=\frac{2}{5},\ t\right)$$

#54: $$t=\frac{15}{4}-\frac{\sqrt{145}}{4} \lor t=\frac{\sqrt{145}}{4}+\frac{15}{4}$$

#55: $$\frac{\sqrt{145}}{4}+\frac{15}{4}-\left(\frac{15}{4}-\frac{\sqrt{145}}{4}\right)$$

#56: $$\frac{\sqrt{145}}{2}$$

d. i. Referring to the graph, we must have a = √5, the x-coordinate of the local maximum.
d. ii. Best done by hand. Alternatively, could find the inverse and then plot.
d. iii. To find the inverse, set y=x(t) and solve for t.

#57: y = x(t)

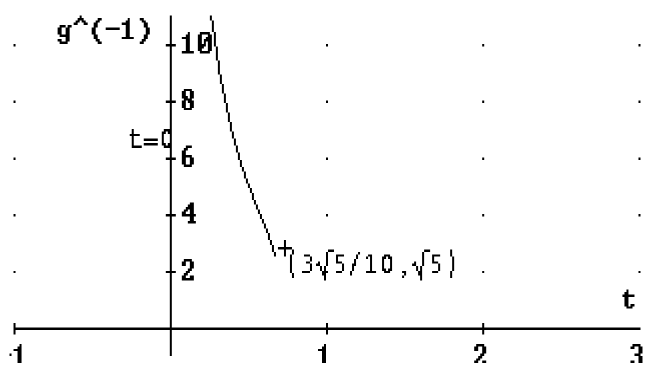
#58: $$y = \frac{3 \cdot t}{t^2 + 5}$$

#59: $$\text{SOLVE}\left(y = \frac{3 \cdot t}{t^2 + 5}, t\right)$$

#60: $$t = \frac{3 - \sqrt{(9 - 20 \cdot y^2)}}{2 \cdot y} \vee t = \frac{\sqrt{(9 - 20 \cdot y^2)} + 3}{2 \cdot y}$$

There are two possible rules to choose from. We want one that will give the larger values of t, so choose the second. Write the answer in correct form, and then obtain a graph.

#61: $$g^{-1} = \frac{\sqrt{(9 - 20 \cdot t^2)} + 3}{2 \cdot t}$$



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