Implicit Differentiation

ID: 8969

Time required
45 minutes

Activity Overview

*Students will be introduced to the concept of implicit differentiation. Students will start by solving a relation for y and using methods for which they are already familiar to find the derivative of this relation. Students will then use the **impDif** command to find an alternate form of the relation. Students will verify that these two forms are equal. Students will also learn how to perform implicit differentiation by hand, using the **impDif** command to check their results. Students will be asked to find the numerical derivative of a relation for specific x-values. A graphical connection to the results found using the implicit differentiation will also be made.*

Topic: Formal Differentiation

- *Use implicit differentiation to find the derivative of a function defined implicitly.*
- *Use the command **impDif** to check the manual computation of implicit differentiation.*

Teacher Preparation and Notes

This investigation offers an opportunity to introduce the concept of implicit differentiation.

- *Students should be familiar with finding derivatives of functions where y is explicitly defined in terms of x. Students should also be familiar with the Chain Rule.*
- *This activity is designed to be **student-centered** with the teacher acting as a facilitator while students work cooperatively. The student worksheet is intended to guide students through the main ideas of the activity and provide a place for them to record their observations.*
- *Reference the Chain Rule to emphasize that y' (or $\frac{dy}{dx}$) must accompany taking the derivative of expressions containing y*
- *Provide students with additional practice finding derivatives using implicit differentiation. Include examples such as sin(2x – 7xy) = 16y that require using the chain rule, product rule, etc.…*
- *Before starting this activity, students should go to the home screen and select **F6: Clean Up > 2: NewProb**, then press* [ENTER]*. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.*
- ***To download the student worksheet, go to education.ti.com/exchange and enter "8969" in the keyword search box.***

Associated Materials

- *ImplicitDifferentiation_Student.doc*

Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

- *Chain Rule (TI-Nspire technology) — 11363*
- *What's the Differential, Dr. Implicit? (TI-Nspire technology) — 11581*

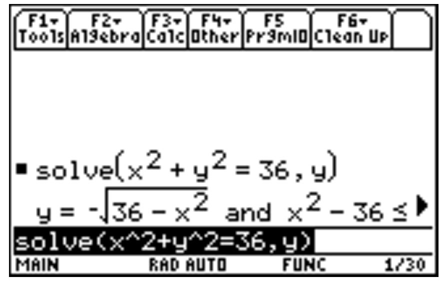
One focus question defines this activity: *How can you find the derivative of a relation, F(x,y) that is not solved for y?*

Use the circle $x^2 + y^2 = 36$ as a discussion point for this focus question. Encourage students to realize that $x^2 + y^2 = 36$ can be solved for *y*, giving two equations—one that defines the top semicircle and one that defines the bottom semicircle. Have them graph both equations on the same set of axes.

Explain to the students that they will in fact solve $x^2 + y^2 = 36$ for *y* and use the resulting equations to find the derivative of $x^2 + y^2 = 36$.

Problem 1 – Finding the derivative of $x^2 + y^2 = 36$

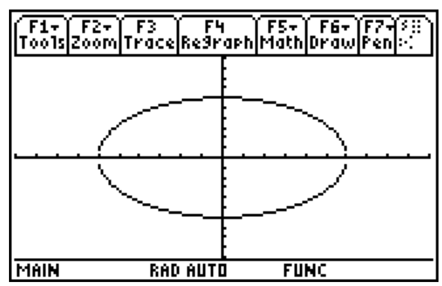
Step 1: Students will solve $x^2 + y^2 = 36$, for *y*. They should be able to this by hand. However, if they use the **Solve** command, **F2:Algebra > 1:Solve**, they will get the two functions with the restrictions on the variables: $f_1(x) = \sqrt{36 - x^2}$ and $f_2(x) = -\sqrt{36 - x^2}$.



Substituting these equations in the original relation yields $x^2 + \left(\sqrt{36 - x^2}\right)^2 = 36$ and $x^2 + \left(-\sqrt{36 - x^2}\right)^2 = 36$. These equations can be simplified as follows:

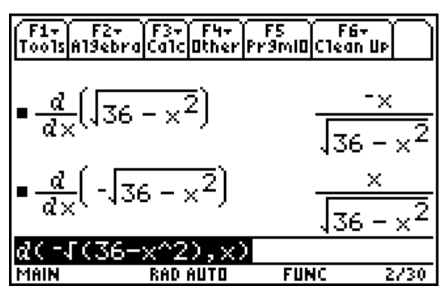
$x^2 + \left(\sqrt{36 - x^2}\right)^2 = 36$	and	$x^2 + \left(-\sqrt{36 - x^2}\right)^2 = 36$
$x^2 + (36 - x^2) = 36$		$x^2 + (36 - x^2) = 36$
$36 = 36$		$36 = 36$

Now students are to graph both functions on the same set of axes and get the circle at the right. Due to the scale of the graphing screen, the circle may be distorted. Students can select **F2:Zoom > 5:ZoomSqr** to adjust the scale. Discuss what problems they might have if they want to find the derivative at $x = 2$.



Step 2: The derivatives of $f_1(x)$ and $f_2(x)$ are:
$\frac{d}{dx} f_1(x) = \frac{-x}{\sqrt{36 - x^2}}$ and $\frac{d}{dx} f_2(x) = \frac{x}{\sqrt{36 - x^2}}$.

The students should be able to do this by hand. If they choose to use the TI-89 Titanium graphing calculator they will get the screen at the right. The keystrokes are **F3:calc>1:*d*(differentiate** or press [2nd] [*d*].

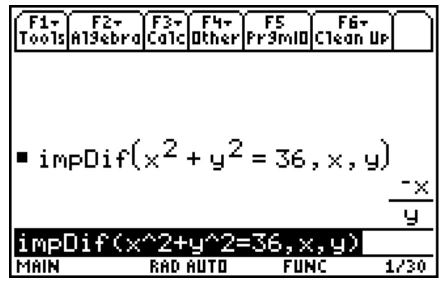


Using these derivatives, the slopes at $x = 2$ are found to be:

$$\frac{-2}{\sqrt{36-2^2}} = -\frac{\sqrt{2}}{4} \approx -0.354 \quad \text{and} \quad \frac{-(-2)}{\sqrt{36-(-2)^2}} = \frac{\sqrt{2}}{4} \approx 0.354 .$$

Students should be encouraged to look at the graph to see if these slopes make sense. They will see that the slopes for any value of x should be equal in magnitude and opposites for each other.

Step 3: Students will use the **impDif** command, shown at the right, to find an alternate form for the derivative of $x^2 + y^2 = 36$. To access the command press **F3:calc > D:impDif(**, where the independent variable is listed first and the dependent variable is listed second.



Students will use $\frac{dy}{dx} = \frac{-x}{y}$ to find the slope of the tangents to $x^2 + y^2 = 36$. First they will need to solve for y, knowing that $x = 2$: $(2)^2 + y^2 = 36 \rightarrow y^2 = 32 \rightarrow y = \pm 4\sqrt{2}$

Substituting $(2, 4\sqrt{2})$ and $(2, -4\sqrt{2})$ into the formula for the derivative yields

$$\frac{dy}{dx} = \frac{-2}{4\sqrt{2}} = -\frac{\sqrt{2}}{4} \approx -0.354 \text{ and } \frac{dy}{dx} = \frac{-2}{-4\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354 .$$

This result is consistent with what was found in Step 2.

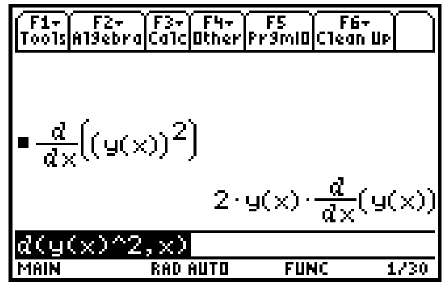
Step 4: It can be shown that the derivatives of $f_1(x)$ and $f_2(x)$ that were found earlier are equal to the result found using the **impDif** command by rewriting the formula $\frac{dy}{dx} = \frac{-x}{y}$ strictly in terms of x:

$$\frac{dy}{dx} = \frac{-x}{y} = \frac{-x}{\sqrt{36-x^2}} \leftarrow \frac{d}{dx}\left(f_1(x)\right) \text{ and}$$

$$\frac{dy}{dx} = \frac{-x}{y} = \frac{-x}{-\sqrt{36-x^2}} = \frac{x}{\sqrt{36-x^2}} \leftarrow \frac{d}{dx}\left(f_2(x)\right).$$

Problem 2 – Finding the Derivative of $x^2 + y^2 = 36$ by Hand

Step 1: Take the derivative of both sides of $x^2 + y^2 = 36$:



$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(36)$$
$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(36)$$
$$2x + \frac{d}{dx}\left(y^2\right) = 0$$

The expression $\frac{d}{dx}\left(y^2\right)$ is evaluated using the **Derivative** command on the calculator screen as shown. Note that we use *y(x)* instead of y.

This is a very important step when using implicit differentiation. Explain to the students that $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx} = 2yy'$ and this result can be justified using the chain rule:

$$y^2 = \left(f(x)\right)^2 \rightarrow \frac{d}{dx}y^2 = \frac{d}{dx}\left(f(x)\right)^2$$
$$= 2f(x)f'(x)$$
$$= 2yy'$$

Step 2: Students can now finish finding the derivative using implicit differentiation:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(36)$$
$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(36)$$
$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

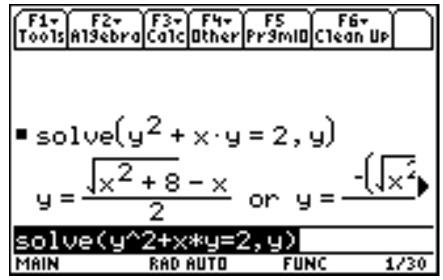
$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-x}{y}$$

This is the same result as was obtained using the **impDif** command.

Problem 3 – Finding the Derivative of $y^2 + xy = 2$

Step 1: The relation $y^2 + xy = 2$ can be solved for y using the method of completing the square. Alternatively, the **solve** command can be used to find two functions, $f_1(x)$ and $f_2(x)$, that *explicitly* define $y^2 + xy = 2$. These are

$$f_1(x) = \frac{\sqrt{x^2+8} - x}{2} \text{ and } f_2(x) = \frac{-\left(\sqrt{x^2+8} + x\right)}{2}$$



While the derivatives of $f_1(x)$ and $f_2(x)$ can be found using methods learned earlier, it should be stressed that implicit differentiation provides a more convenient method.

Step 2: Using implicit differentiation, the derivative of $y^2 + xy = 2$ can be found by hand:

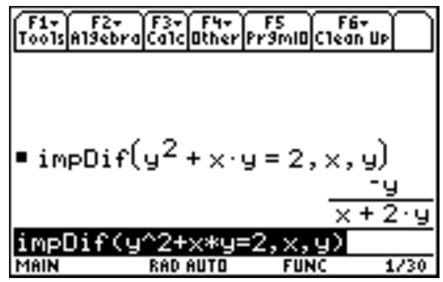
$$\frac{d}{dx}y^2 + \boxed{\frac{d}{dx}xy} = \frac{d}{dx}2$$

$$2yy' + \boxed{xy' + y} = 0$$

$$2yy' + xy' = -y$$

$$(2y + x)y' = -y$$

$$\frac{dy}{dx} = \frac{-y}{2y + x}$$



Check with the students to make sure that they use the product rule to find the derivative of xy:

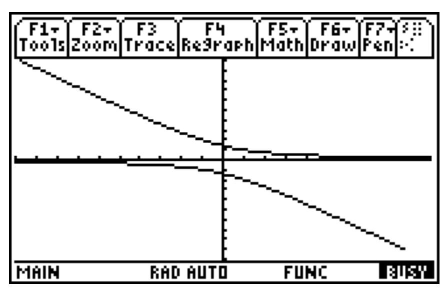
$$\frac{d}{dx}(xy) = x \bullet \frac{d}{dx}y + y \bullet \frac{d}{dx}x$$

$$= xy' + y$$

Students will confirm their answer by using the **impDif** command as shown to the right.

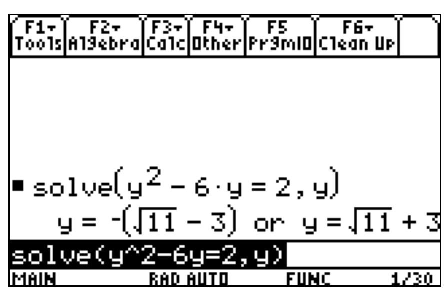
Step 3: To check this result graphically, students should first graph the two functions:

$$f_1(x) = \frac{\sqrt{x^2+8} - x}{2} \text{ and } f_2(x) = \frac{-\left(\sqrt{x^2+8} + x\right)}{2}$$

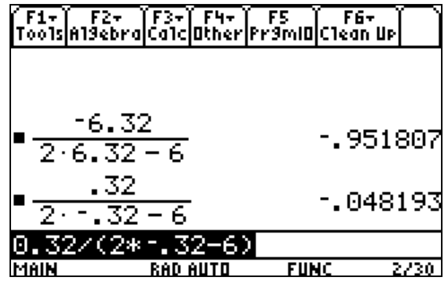


If they let $x = -6$, then they can solve to find the y-value. Students will find two values $3 + \sqrt{11} \approx 6.32$ and $3 - \sqrt{11} \approx -0.32$. Thus, there are two points (–6, 6.32) and (–6, –0.32).

When they use those values in the implicit derivative $\frac{dy}{dx} = \frac{-y}{2y + x}$, they will get the slope of –0.95 for (–6, 6.32) and the slope of –0.05 for (–6, –0.32).



Students can pick another *x*-value (e.g., *x* = 1) and do a similar analysis.



Extension – Finding the Derivative of $x^3 + y^3 = 6xy$

Step 1: The relation $x^3 + y^3 = 6xy$ cannot be solved explicitly for *y.* In this case implicit differentiation must be used.

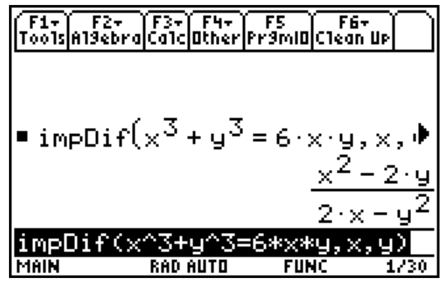
$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}6xy$$

$$3x^2 + 3y^2y' = 6xy' + 6y$$

$$3y^2y' - 6xy' = 6y - 3x^2$$

$$\left(3y^2 - 6x\right)y' = 6y - 3x^2$$

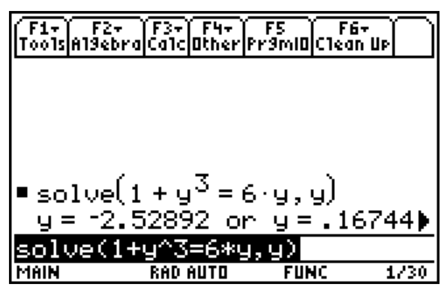
$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$



The **impDif** command is used to confirm this result. Students may need to do the work to see the two solutions are the same.

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x} = \frac{-x^2 + 2y}{-2x + y^2} = \frac{-(x^2 - 2y)}{-(2x - y^2)} = \frac{x^2 - 2y}{2x - y^2}$$

Step 2: Students will use $\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$ to find the slope of the tangents to $x^3 + y^3 = 6xy$ at *x* = 1. Using the **solve** command, students will find that the ordered pairs with *x*-coordinate equal to 1 are (1, –2.529), (1, 0.167) and (1, 2.361).



The slopes of the tangents can now be calculated on the *Calculator* screen as shown. Students will get the following results:

(1, –2.52892) *m* = –1.28782
(1, 0.16449) *m* = 0.33728
(1, 2.36147) *m* = 1.04093