EXPLORATIONS

Chapter 9

Space Trajectories

In this chapter, you will explore different models for the motion of satellites. You will be working with two-and three-body problems, and considering both artificial and natural satellites.

Introduction

Consider first the two-body problem. Newton's Law of Universal Gravitation (1687) states that any two bodies attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them.

Assume that one of the bodies (the earth, for example) has mass much larger than the other (a communication satellite, for example). This allows you to assume a coordinate system with the origin at the center of the larger body and with only two coordinate directions (giving the plane of the orbit of the smaller body about the larger one). When the two bodies are more nearly the same size, they both rotate about their combined center of mass.

For simplicity, also assume that our coordinate system gives an inertial frame of reference. This is not exactly true, because the earth rotates about the sun, and because the sun and our galaxy move relative to other galaxies, but it is an acceptable first approximation.

Then Newton's law becomes

$$\mathbf{F} = -\frac{G\,M\,m}{r^2}\frac{\mathbf{r}}{r}, \quad \text{or} \quad \begin{bmatrix} F_x \\ F_y \end{bmatrix} = -\frac{G\,M\,m}{x^2+y^2}\begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2}} \end{bmatrix},$$

where **F** is the force on the mass m due to the gravitational attraction of mass M, G is the universal gravitational constant, and

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$$

is the position vector for the location of mass m with length

$$r = \sqrt{x^2 + y^2} \ .$$

Assume that there are no other forces acting on the mass m (i.e. no rocket booster, no maneuvering thrustors, or no atmospherical drag). Newton's Second Law of Motion tells us that

$$\mathbf{F} = m\,\mathbf{a} \quad \text{or} \quad m\begin{bmatrix}\ddot{x} \\ \ddot{y}\end{bmatrix} = -\frac{G\ M\ m}{\left(x^2 + y^2\right)^{3/2}}\begin{bmatrix}x \\ y\end{bmatrix}.$$

Note that this vector equation is equivalent to a system of second-order equations in x and y.

Of course, *Kepler's Laws* (1609-1619) tell us how the solution should turn out.

- The orbit of mass m is an ellipse, with mass M at a focus.
- The line joining mass m to mass M sweeps out equal areas in equal time.
- The square of the period of the mass m is proportional to the cube of its mean distance to mass M.

The gravitational constant G is a built-in constant on the TI-86. Press [2nd] [CONS] [F1] (**BLTIN**) [MORE] to display **Gc** in the CONS BLTIN (Built-In Constants) menu. **Gc** = 6.67259E-11 has units Newton-meter^2/kilogram^2 (N-m^2/kg^2). For simplicity, you store this constant in the single-letter variable **G**. To use this gravitational constant, you need to measure mass in kilograms (kg), length in meters (m), and time in seconds (s) since a Newton is defined to be N=1kg-m/s^2. For our purposes, it is sufficient to know that the mass of the earth
M_e = 5.9737E24 kg. Store this mass in the variable **M**. (You cannot use **Me** which is a built-in constant for the mass of an electron.)

From *Bate: Fundamentals of Astrodynamics*, you take two useful formulas. All of these can be derived from the differential equations for the two-body problem, but it takes several chapters in that book to do this! These formulas can also be found in *Escobal* or *Wertz*. For circular orbits, the period T is given in terms of the radius r by

$$T = \frac{2\pi}{\sqrt{G\ M_e}}\, r^{\frac{3}{2}}.$$

(Note the agreement with Kepler's third law.) The speed v (which is the magnitude of the velocity vector $\mathbf{v} = \mathbf{r}'$) for a circular orbit is given by

$$v = \sqrt{\frac{G\ M_e}{r}}\ .$$

A satellite is said to be in a *geosynchronous orbit* if it rotates about the earth in a circular orbit with a period exactly the same time as the earth rotates about its axis (and you usually also ask that it move in the same direction around). Thus if a satellite in geosynchronous orbit is directly above the earth's equator, the satellite would appear fixed in the same location in the sky. Many communication satellites are placed in a geosynchronous orbit so that a transmitter or receiver (such as a home satellite dish) can be aimed toward this fixed location.

Example 1: A Geosynchronous Orbit

Plot a geosynchronous orbit by determining appropriate initial conditions for the two-body model.

Solution

You need

$$T = 1 \text{ day} = \left(60\ \frac{\text{sec}}{\text{min}}\right)\left(60\ \frac{\text{min}}{\text{hour}}\right)\left(24\ \frac{\text{hour}}{\text{day}}\right)(1 \text{ day}) = 86{,}400 \text{ sec}$$

so that the satellite revolves around the earth exactly as the earth is rotating (a geosynchronous orbit).

1. Store this period in variable **T** on your TI-86.
2. Solving for the radius r you find

   $$r = \left(\frac{T\sqrt{G\ M_e}}{2\pi}\right)^{\frac{2}{3}} = 42{,}241{,}098.0421 \text{ m}$$

   which you store in the variable **R**.
3. The needed speed v is given by

   $$v = \sqrt{\frac{G\ M_e}{r}} = 3{,}071.85933538\ \frac{\text{m}}{\text{s}}$$

   which you store in **V**.

   Of course you do not know r and v to this accuracy because you do not know G and M_e this accurately, but it is easiest to do these computations and store the results with no rounding.

   Note: *Keep these values stored in T, R, and V throughout this chapter so that you can use them in several examples and exercises.*
4. Convert the second order system of two equations in two unknown functions into a first order system in four unknowns in the standard way.

   $x(t) = Q1(t)$ Q'1(t) = Q2

   $\dot{x}(t) = Q2(t)$ Q'2(t)=–G*M*Q1/((Q1²+Q*3²)^(3/2))

   $y(t) = Q3(t)$ Q'3(t)=Q4

   $\dot{y}(t) = Q4(t)$ Q'4(t)=–G*M*Q3/((Q1²+Q*3²)^(3/2))
5. Assuming that you expect a circular orbit about the origin, you choose initial conditions where the orbit crosses the positive x-axis (x = R, y = 0, t = 0). You expect the initial velocity vector to be tangent to the circle, hence perpendicular to the x-axis. To go counter clockwise this would mean $\dot{x}(0) = 0$ and $\dot{y}(0) = V$ (straight upward in the xy-plane).

   Thus you set the initial conditions editor (**INITC**) to **tMin** = 0, **QI1** = R, **QI2** = 0, **QI3** = 0, and **QI4** = V.
6. Change the style to **Path** (-0) so that you can see the plotting position as it moves two revolutions.

7. Set the window parameters to

   tMin = 0, **tMax** = 2*T, **tStep** = T/100 **tPlot** = 0, **xMin** = -R,

   xMax = R, **xScl** = R/10, **yMin** = -R, **yMax** = R, **yScl** = R/10,

   difTol = 0.005 (making sure that the **RK** computational method is selected in the GRAPH FORMAT menu).

9. Set the **AXES** to **x** = Q1 and **y** = Q3.

10. To make the plot look like a circle, press [F3] (**ZOOM**) [MORE] [F2] (**ZSQR**) to have the graph drawn with equally scaled axes. (Figure 9.1)

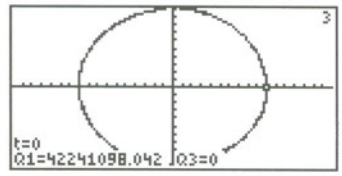

Figure 9.1

Example 2: A Shuttle Orbit

In May 1997, the shuttle Atlantis on mission STS-84 docked with the Russian space station MIR. Checking the orbit parameters on NASA's web pages (http://shuttle.nasa.gov/current/orbit/orbiter/sighting/shuttle.html), you find that the height above the earth's surface of the docked spacecraft was about 213 nautical miles or 394476 m (note the [CONV] feature of the TI-86).

Assuming a circular orbit, find the radius and the velocity needed for such a low earth orbit. Modify the initial conditions (and the window) from Example 1 to show one period of such an orbit. Then from the home screen, use the **CIRCL** (0, 0, 6378145) to draw in a circle representing the earth's surface since the mean equatorial radius of the earth is 6,378,145 m.

Solution

1. Using the same formulas as in Example 1, compute that r = 6772621 m (*height + earth's radius*), T = 5546.84593399 s (about 92.45 minutes), and v = 7,671.68103544 m/s.

   Note: *Do not delete or overwrite the variables T, R, and V from Example 1 because you will need them later. Instead, store the new quantities in TT, RR, and VV.*

2. Retain essentially the same settings as in Example 1. Simply make the corresponding changes from **T** to **TT**, **R** to **RR**, and **V** to **VV** in the window settings and the initial conditions.

3. After the orbit is plotted, give the command **CIRCL** (0, 0, 6378145) on the home screen ([2nd] [CATLG-VARS] [F1] (**CATLG**)) to draw the surface of the earth. Note how close the orbit of the satellite is to the surface of the earth in Figure 9.2.

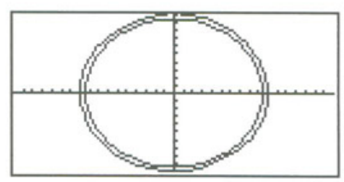

Figure 9.2

A very common maneuver in space is to move from one nearly circular orbit to another. For example, the shuttle might release a communication satellite that needs to move out to a geosynchronous orbit. The most fuel efficient method for doing this is called a *Hohmann transfer orbit*. The transfer orbit is an ellipse that is tangent to each of the circles.

Theoretically, one would only need to go half a revolution on the Hohmann orbit to transfer from one circle to the next. In practice, the satellite will often travel several revolutions in the Hohmann transfer orbit while very accurate measurements are made of the actual resulting orbit (which may not be exactly the one planned and desired).

The final maneuver into the geosynchronous orbit can be adjusted slightly to reflect the actual transfer orbit. Suppose the radii of the two circular orbits satisfy $r_1 < r_2$ where you want to move from the smaller to the larger circle. Then

$$v_1 = \sqrt{\frac{G\ M_e}{r_1}} \quad \text{and} \quad v_2 = \sqrt{\frac{G\ M_e}{r_2}}$$

give the needed velocities for the two circular orbits.

From *Bate* (p. 163-165) or *Wertz* (p. 56-57), the change in velocity needed to move out of the smaller circular orbit into the elliptical transfer orbit is

$$\Delta v_1 = \sqrt{G\ M}\left\{\sqrt{\frac{2}{r_1} - \frac{2}{r_1 + r_2}} - \sqrt{\frac{1}{r_1}}\right\},$$

and the change in velocity needed to move out of the elliptical transfer orbit into the larger circular orbit (where the change must occur when the spacecraft is in the location on the transfer orbit tangential to the larger circular orbit) is

$$\Delta v_2 = -\sqrt{G\ M}\left\{\sqrt{\frac{2}{r_2} - \frac{2}{r_1 + r_2}} - \sqrt{\frac{1}{r_2}}\right\}.$$

The total period for the transfer orbit T_h (one complete revolution, so that you can plot it) is given by

$$T_h = \frac{2\pi\left(\frac{r_1 + r_2}{2}\right)^{\frac{3}{2}}}{\sqrt{G\ M}}.$$

A typical figure showing the two circular orbits (plot circles) and the Hohmann transfer orbit is shown in Figure 9.3.

In this figure, the axes have been turned off, the center and the two "transfer points" are marked by "boxes" using [2nd] [STAT] [F3] (**PLOT**), and the two circular orbits were drawn using the **CIRCL** command.

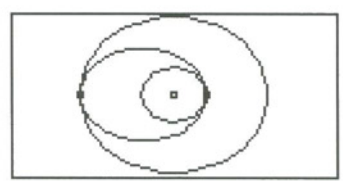

Figure 9.3

Example 3: Hohmann Transfer Orbit

Calculate the parameters and plot the Hohmann transfer orbit needed to go from a low-earth shuttle orbit (as in Example 2) to a geosynchronous orbit (Example 1). Use the **CIRCL** command to plot the two circular orbits.

Solution

1. From Example 1, you have calculated the period time **T**, the radius r_2 = **R**, and the velocity v_2 = **V** for the geosynchronous orbit. (Figure 9.4)

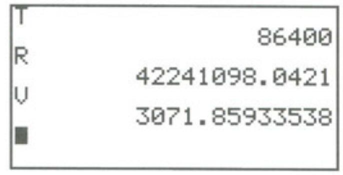

Figure 9.4

2. You can also use the x- and y- ranges from that example to be able to just view the largest circle in an equally-scaled plot. (Figure 9.5)

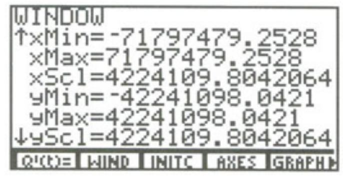

Figure 9.5

3. From Example 2, you have calculated the period time **TT**, the radius r_1 = **RR**, and the velocity v_1 = **VV** for a typical shuttle orbit. (Figure 9.6)

Figure 9.6

4. The two changes in velocity needed (by a rocket booster) are

   $\Delta v_1 = 2{,}400.28625684 \text{ m/s}$ and $\Delta v_2 = 1456.99560416 \text{ m/s}$.

   The actual keystrokes for computing these values and storing them in **DV1** and **DV2** are shown in Figure 9.7.

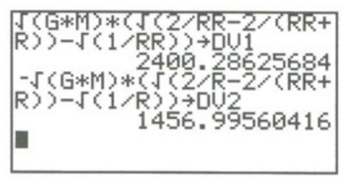

Figure 9.7

5. Compute and store the total period time for the transfer orbit to **TH**. The result is 38,180.5724818 s (with the minimal transfer time half of this or about 5.3 hours), as shown in Figure 9.8.

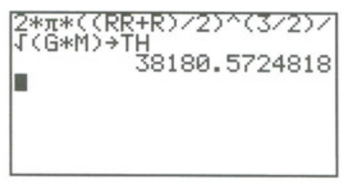

Figure 9.8

6. Finally, use the differential equation plot only for the transfer orbit by plotting the circles with the **CIRCL** command as in Figure 9.9.

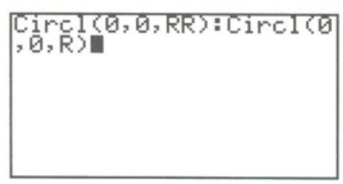

Figure 9.9

The final plot is shown in Figure 9.10.

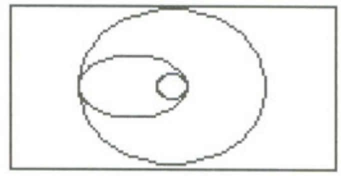

Figure 9.10

Finally, briefly consider a restricted 3-body problem. Your simplified development roughly follows that of *Borse: Numerical Methods with MATLAB*, with more terms included.

Suppose that a large body of mass M (the earth 5.9737 E24 kg, for example) and a moderately large body of mass m (the moon 7.34765 E22 kg, for example) are each moving in a circular orbit about their common center of mass with period P (which for the earth's moon is about 27.3 days).

You will choose a *noninertial* coordinate system XY where the origin is the center of the larger body and where the smaller body remains in a fixed location on the positive x-axis. Let D be the distance between the center of M and the center of m (for the moon, D = 3.84400 E8 m).

The third body (a research satellite, for example) is assumed to have mass so small that it does not affect the first and second bodies. Thus you need only analyze the forces acting on the third body. These are the gravitational attraction toward mass M and the gravitational attraction toward mass m. The angular speed of rotation

$$\omega = \frac{2\pi}{P} = \frac{\sqrt{G\,(M+m)}}{D^{\frac{3}{2}}}$$

describes how all three bodies also rotate in an inertial system about the common center of mass (located at the point

$$X = \frac{m}{M+m}D\,,\ Y = 0$$

where the third body is ignored). The equations become

$$\ddot{X} = -\frac{G\,M\,X}{\left(X^2+Y^2\right)^{\frac{3}{2}}} - \frac{G\,m\,(X-D)}{\left((X-D)^2+Y^2\right)^{\frac{3}{2}}} - 2\,\omega\,\dot{Y} + \omega^2\left(X - \frac{m\,D}{M+m}\right)$$

$$\ddot{Y} = -\frac{G\,M\,Y}{\left(X^2+Y^2\right)^{\frac{3}{2}}} - \frac{G\,m\,Y}{\left((X-D)^2+Y^2\right)^{\frac{3}{2}}} + 2\,\omega\,\dot{X} + \omega^2\,Y$$

It turns out that there are exactly five locations, called *Lagrange points* (or libration points) in the XY plane which are stationary points for this system. Thus a satellite at rest exactly at these positions will stay there (rotating just as M and m do about the common center). You will get some very interesting trajectories simply by being very close to these positions with small velocities.

These points have been proposed as possible locations for a permanent space station (although the stable ones are now located too far from earth for convenient servicing). In the Sun-Jupiter system, one of these Lagrange points is the location of the Trojan asteroid group. You also get interesting trajectories when you attempt to "loop" around both M and m as in a trip to the moon.

Example 4: An Orbit Near a Lagrange Point

Verify that $X = D/2$, $Y = \sqrt{3}\,D/2$, $\dot{X} = 0$, and $\dot{Y} = 0$ is a stationary point for this system (this Lagrange point is labeled L_4). Implement this system in the differential equation editor (**Q'(t)=** screen) for the earth-moon and investigate the trajectory using initial conditions $X = D/2$, $Y = \sqrt{3}\,D/2$, $\dot{X} = 0$ m/s, and $\dot{Y} = -10$ m/s.

Solution

You have stored G = 6.67259 E-11 and M = 5.9737 E24. Add the mass of the moon m with MM = 7.34765 E22, the distance between the centers of the earth and moon D = 3.84400 E8, the combined mass center

$$\text{C} = \frac{m}{M+m}D = 4670670.12183,$$

and the angular speed

$$\text{A} = \frac{\sqrt{G\,(M+m)}}{D^{\frac{3}{2}}} = 2.66531457348\text{ E-6},$$

all of which are in the consistent set of units for our equations.

Suggestion: *Since many of the exercises use the system of differential equations in the previous exercises, you might want to save that system in a graphical database (GDB) using the* [GRAPH] [MORE] [MORE] [F2] (**STGDB**) *command before entering the new equations.*

1. Type in the system:

$$\begin{aligned} &Q'1 = Q2 \\ &Q'2 = \text{-}G*M*Q1/((Q1^2+Q3^2)\wedge(3/2)) - G*MM*(Q1-D)/(((Q1-D)^2+Q3^2)\wedge(3/2)) \\ &\qquad -2*A*Q4 + A^2*(Q1-C) \\ &Q'3 = Q4 \\ &Q'4 = \text{-}G*M*Q3/((Q1^2+Q3^2)\wedge(3/2)) - G*MM*Q3/(((Q1-D)^2+Q3^2)\wedge(3/2)) \\ &\qquad +2*A*Q2 + A^2*Q3 \end{aligned}$$

   It is straight-forward (but tedious) to substitute the initial conditions **QI1**=D/2, **QI2**=0, **QI3**=$\sqrt{3}$ *D/2, and **QI4**=0 into the equations to verify that it is a stationary point (that is, these values make the right-hand sides zero, although rounding might cause the TI-86 to deviate from this slightly).

2. Instead, change to the initial conditions **QI1**=D/2, **QI2**=20, **QI3**=$\sqrt{3}$ D/2, and **QI4**=-10 indicating that you start at this Lagrange point but with non-zero velocity.

3. Select **AXES** to be **x**=Q1, **y**=Q3, and **WIND** to begin **tMin**=0, **tMax** = 60*T, **tStep**=5000, **tPlot**=0, **difTol**=.1.

4. Press [MORE] [F3] (**ZOOM**) [MORE] [F1] (**ZFIT**) to see just the resulting trajectory. The resulting plot is given in Figure 9.11 with the approximate *XY* window given below. The starting point is the end closest to the middle of the plot.

   1.60515 E8 < X < 2.02301 E8

   3.17433 E8 < Y < 3.52327 E8

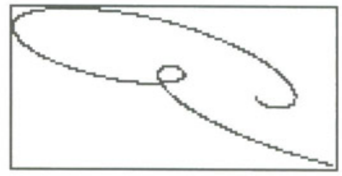

Figure 9.11

The Lagrange point L_5 is given by the initial conditions $X = D/2$, $Y = -\sqrt{3}\, D/2$, $\dot{X} = 0$, and $\dot{Y} = 0$. The Lagrange points L_1, L_2, and L_3 lie on the X axis, separated by the locations of the earth and moon centers. You will explore these in the exercises.

Example 5: A Trip From Near Earth to Near the Moon

Show both the location of the earth and the moon in a viewing window without axes, and plot the restricted three-body trajectory with initial conditions $X = -2.7$ E7 m, $\dot{X} = 0$, $Y = 0$, and $\dot{Y} = -5450$ m/s for time $0 < t < 345600$ s = 3 days.

Solution

1. Store the location of the earth (0, 0) and the location of the moon (D, 0) in two lists, and turn on a statistical scatter plot to mark the location of the two larger bodies. (Figures 9.12 and 9.13)

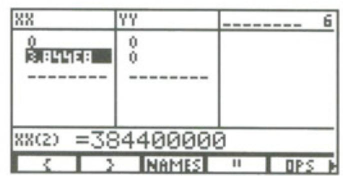

Figure 9.12

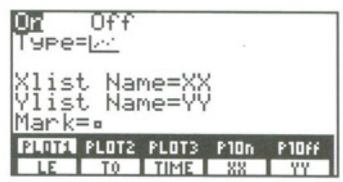

Figure 9.13

2. Set the viewing window to **xMin** = $-D/2$, **xMax** = $3D/2$, **yMin** = $-D/2$, **yMax** = $D/2$, with **difTol** = .1 and **tStep**=1000. Set the given initial conditions, and press [MORE] [F3] (**ZOOM**) [MORE] [F2] (**ZSQR**). (Figure 9.14)

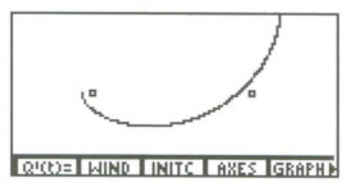

Figure 9.14

Exercises

1. Find the orbit parameters and plot the two-body model for LANDSAT-4, a low-earth satellite with a period of 99 minutes.

2. Suppose that a shuttle mission is in a circular orbit with a height of 800 nautical miles above the earth's surface. Find the needed velocity changes for a Hohmann transfer orbit to a lower circular orbit with a height of 180 nautical miles.

   Note: *You will need to make the changes with opposite signs to go from the higher to the lower orbit compared to the formulas in the text going from lower to higher. Also you can use [CONV] operations to convert nautical miles to meters.*

3. The rotation period for the moon is 27.321661 days. Begin with a circular model. The mean earth-moon distance is 384400 km, so you will use this as the radius for a circular orbit. Compute the period and other orbit parameters for this radius.

   In fact, the orbit is not circular but ranges from 356400 to 406700 km. Use a Hohmann transfer orbit between two circular orbits with these two values to get an elliptical model. Determine the period and plot this Hohmann transfer orbit.

4. In the restricted three-body model for the earth-moon system, find the value $X = l_3 < 0$ so that $X = l_3$, $Y = 0$, $\dot{X} = 0$, and $\dot{Y} = 0$ is a stationary point. Take care that you simplify (x^2)^(3/2) to be (abs(x))^3 and not x^3. Note that this reduces to a root-finding problem that you can do in many ways on the TI-86. Explore trajectories near this Lagrange point L_3.

5. In the restricted three-body model for the earth-moon system, find the value $X = l_1$ with $0 < l_1 < D$ so that $X = l_1$, $Y = 0$, $\dot{X} = 0$, and $\dot{Y} = 0$ is a stationary point. Take care that you simplify (x^2)^(3/2) to be (abs(x))^3 and not x^3. Note that this reduces to a root-finding problem which you can do in many ways on the TI-86. Explore trajectories near this Lagrange point L_1.

6. In the restricted three-body model for the earth-moon system, find the value $X = l_2 > D$ so that $X = l_2$, $Y = 0$, $\dot{X} = 0$, and $\dot{Y} = 0$ is a stationary point. Take care that you simplify (x^2)^(3/2) to be (abs(x))^3 and not x^3. Note that this reduces to a root-finding problem which you can do in many ways on the TI-86. Explore trajectories near this Lagrange point L_2.

7. Plot some trajectories in the Sun-Jupiter restricted three-body system, where the mass of the Sun is M = 1.989 E30 kg, the ratio of the mass of the Sun to the mass of Jupiter is M/m = 1.047355 E3, and the mean distance between them is D = 71,420 km. Note that several interplanetary probes have passed near Jupiter and been accelerated for further travel.

8. Try to find initial conditions for a trajectory in the earth-moon restricted three-body problem which will leave near the earth, loop around the moon, and return near the earth.