Ages 17-19 – Elimination of parameters and substitution with computer algebra

1. Introduction

Starting with the geometrical concept of parametric equations of lines and planes, we illustrate the method of elimination to obtain a cartesian equation. This elimination can be done in a direct and simple way, close to the meaning of the elimination process, by using the procedures "solve" and "substitute" of a Computer Algebra System (the basic algebraic manipulations of formulas).

Without CAS, this method is difficult to realize by hand for a plane. Therefore it was necessary to introduce in advance more elegant (but also more sophisticated) algebraic techniques like determinants. The result was that for a lot of students the meaning of the elimination process disappeared behind these algebraic manipulations.

Later on in the educational process, we have the opportunity to show the equivalence and strength of the new algebraic techniques.

These ideas will be illustrated using a few (geometric) examples.

2. A line in space

Start at the point $(2,5,1)$ and take t "steps" of vector $(1,4,9)$ to reach a point (x,y,z). As t runs over $\mathbb{R}$, we can "see" how a line L of points (x,y,z) is generated.

Parametric equations of L are the algebraic translation of the geometric description of L:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \quad \text{or} \quad \begin{cases} x = 2 + t \\ y = 5 + 4t \\ z = 1 + 9t \end{cases} .$$

For $t = 0$ we find the point (2,5,1) and for $t = 2$ the point (4,13,19) on the line.

Alternatively, does the point P(13, 49, 100) lie on L? In other words, can we find a convenient step t in our parametric description to reach the point P? We know P will lie on L if the system

$$\begin{cases} 13 = 2 + t \\ 49 = 5 + 4t \\ 100 = 1 + 9t \end{cases}$$

has a solution for t. We can verify this by solving the first equation for t (this yields t = 11) and then substituting this value into the two last equations to check if the equations are true for the given value of t. These conditions are fulfilled, thus P lies on L.

The same procedure can be followed for an arbitrary point (x,y,z) in space. The point (x,y,z) will lie on the line if the system

$$\begin{cases} x = 2 + t \\ y = 5 + 4t \\ z = 1 + 9t \end{cases}$$

has a solution for t. Solving the first equation for t yields $t = x - 2$.

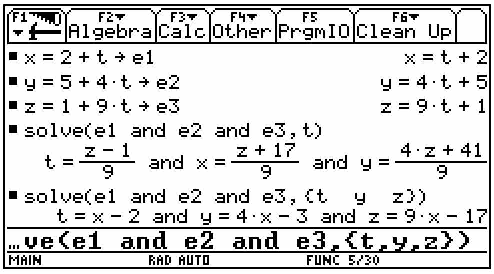
This solution must validate the second and the third equation:

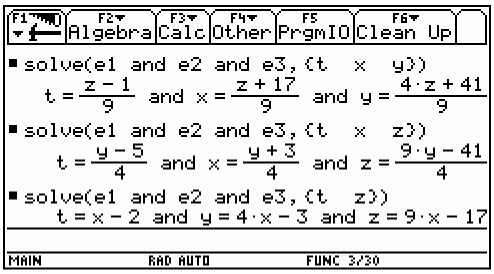
$$\begin{cases} y = 5 + 4(x-2) \\ z = 1 + 9(x-2) \end{cases} \quad \text{or} \quad \begin{cases} y = 4x - 3 \\ z = 9x - 17 \end{cases}$$

We get two conditions for the point (x,y,z), these conditions are the cartesian equations of L; the line is described as the intersection of two planes.

Finding cartesian equations of a line means that we have to find the conditions for the coordinates (x, y, z) of a general point in order that the system would have a solution for t. To do this we *eliminate* the parameter t .

CAS can be used to perform the elimination (but calculation by hand is faster) :





Solving for t (given x, y, z) with the instruction `solve(e1 and e2 and e3,t)` yields the extra *conditions* $x = \dfrac{z+17}{9}$ and $y = \dfrac{4z+41}{9}$. This illustrates the concept of elimination.

Solving for t, y and z (given x) with `solve(e1 and e2 and e3,{t,y,z})` yields the same planes $y = 4x - 3$ and $z = 9x - 17$ which were obtained in our calculation by hand, but we lose the underlying idea of elimination!

3. Cartesian equation of a plane

Starting at a given point, we now have to take steps in two different directions to reach every point of a plane.

Start at the point $(6,9,1)$ and take r "steps" of vector $(-4,12,5)$ followed by s steps of vector $(1,3,5)$ to reach a point (x, y, z). As r and s run over $\mathbb{R}$, we can "see" how a plane α of points (x, y, z) is generated. For integer values of r and s, we reach the grid points of a grid on α defined by the given vectors.

Parametric equations of α are the algebraic translation of the geometric description of α:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix} + r \begin{bmatrix} -4 \\ 12 \\ 5 \end{bmatrix} + s \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{or} \quad \begin{cases} x = 6 - 4r + s \\ y = 9 + 12r + 3s \\ z = 1 + 5r + 5s \end{cases}.$$

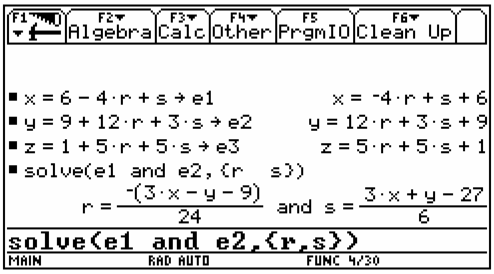
Following the same procedure as for a line, students should be able to investigate whether a given point lies on the plane or not.

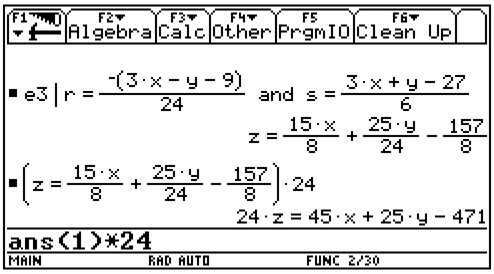
Generalizing that procedure for an arbitrary point (x,y,z) in space, the point (x,y,z) will lie on the plane if the system

$$\begin{cases} x = 6 - 4r + s \\ y = 9 + 12r + 3s \\ z = 1 + 5r + 5s \end{cases}$$

has a solution for r and s.

We now use the two first equations, solve for r and s and then substitute the resultant expressions into the third equation. Where possible get rid of the denominators.

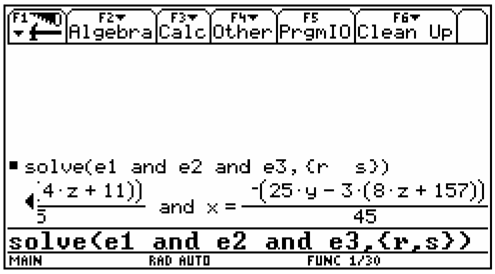




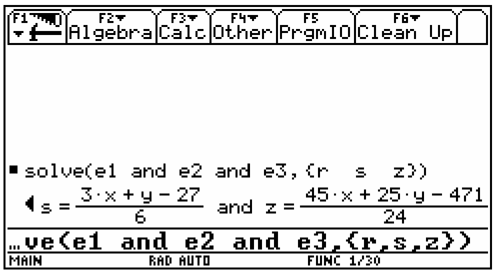
The plane α has the equation $45x + 25y - 24z - 471 = 0$.

This procedure is very close to the meaning of the elimination process.

The elimination can also be done in one CAS-step by solving the three equations for r and s (given x, y, z) using the instruction solve(e1 and e2 and e3, {r , s}). The last equation gives the condition on (x, y, z) for the existence of a solution for r and s.

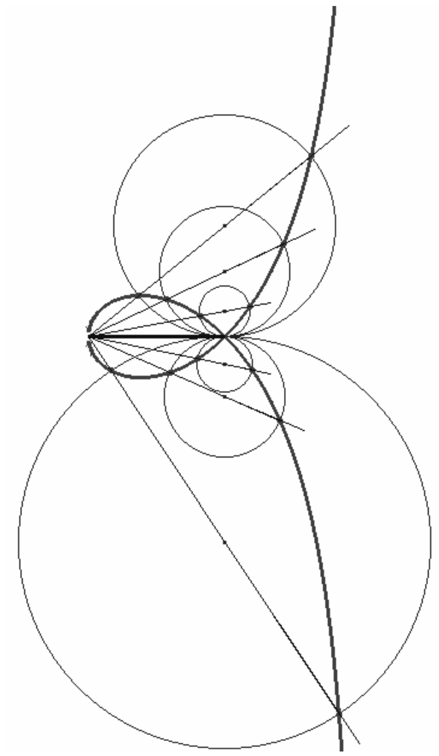


Solving for r, s and z (given x and y) with `solve(e1 and e2 and e3,{r,s,z})` yields the same plane $45x + 25y - 24z - 471 = 0$ but has a different algebraic meaning, losing the idea of elimination.

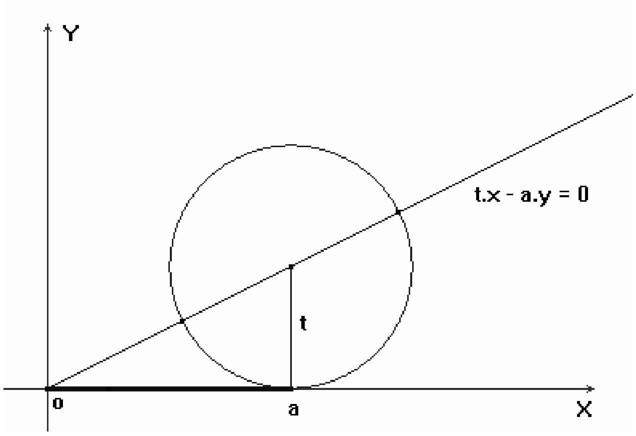


4. The strophoid

Given a line segment with length a, consider a circle tangent to the line segment at one of its endpoints. Draw a line through the other endpoint and the centre of the circle. The locus of the points of intersection of this line with the circle, by allowing the circle to vary, is called a strophoid.



Here we follow the same idea of elimination in the context of plane analytic geometry and the geometric description of loci. Parametric equations are the algebraic translation of the geometric construction process and from these equations we find the cartesian equation by elimination of the parameter(s).

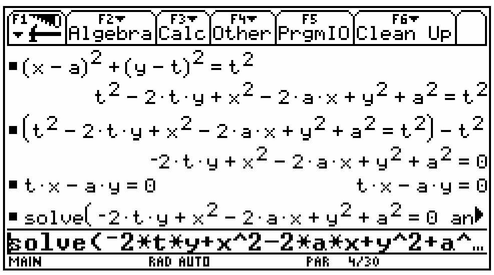


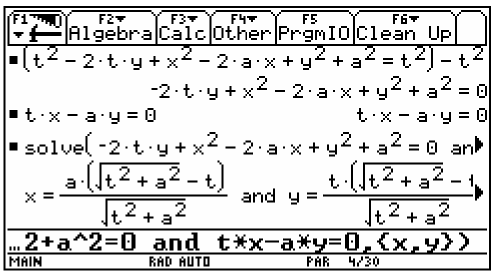
The locus consists of the points of intersection of the associated curves: the line $t \cdot x - a \cdot y = 0$ and the circle $(x-a)^2 + (y-t)^2 = t^2$, when choosing the subjoined coordinate system.

The classical procedure consists of eliminating the parameter *t* directly from the system of equations of the associated curves (see next page), because it is too difficult to find the (algebraically equivalent) parametric equations by hand.

The difficulty of calculation can be overcome when calculations are left to a calculator with CAS. It is more natural to look for the points of intersection of the associated curves which gives the parametric equations of the curve.

We enter the equations of the circle and the line and solve for *x* and *y*:

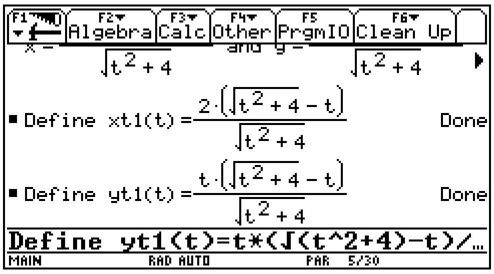


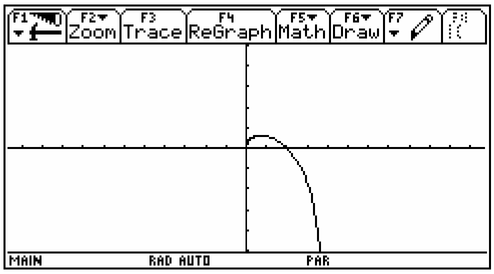


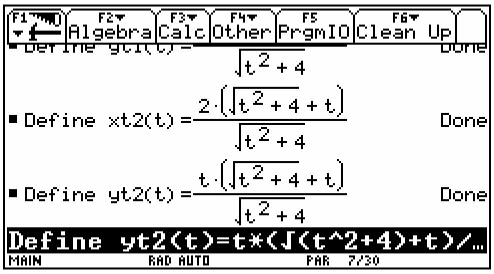
Since the circle and the line have two points of intersection, we find two solutions (connected with "or"):

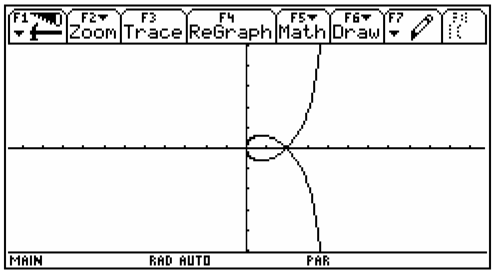
$$x=\frac{a\left(\sqrt{t^2+a^2}-t\right)}{\sqrt{t^2+a^2}} \text{ and } y=\frac{t\left(\sqrt{t^2+a^2}-t\right)}{\sqrt{t^2+a^2}} \quad \text{or} \quad x=\frac{a\left(\sqrt{t^2+a^2}+t\right)}{\sqrt{t^2+a^2}} \text{ and } y=\frac{t\left(\sqrt{t^2+a^2}+t\right)}{\sqrt{t^2+a^2}}$$

Now draw the curve with these parametric equations for $a=2$:

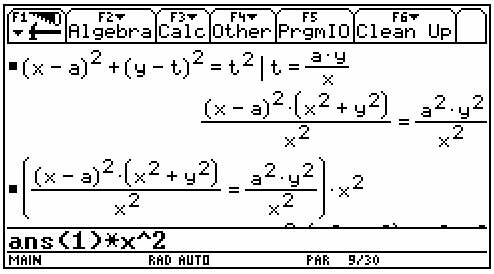


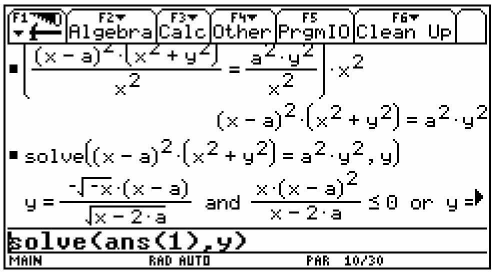




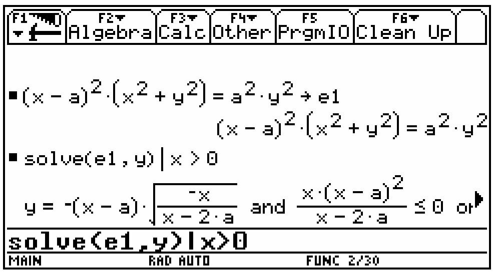


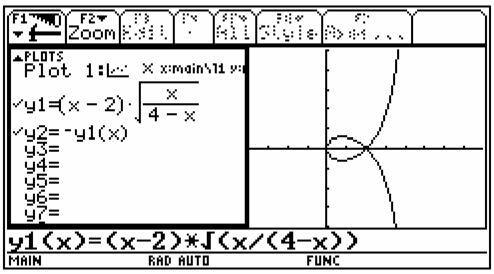
By eliminating the parameter *t* from the system of the associated curves, we expect the cartesian equation of the strophoid.



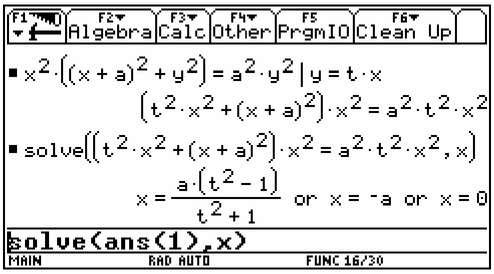


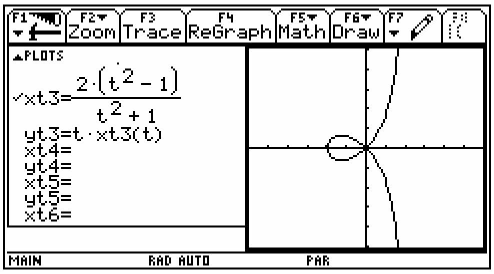
However an incorrect result is obtained, namely the functions $y=\pm\frac{(x-a)\sqrt{-x}}{\sqrt{x-2a}}$, with the null set as domain! By adding the condition $x>0$ we get the correct functions $y=\pm(x-a)\sqrt{\frac{-x}{x-2a}}$, yielding the same curve for $a=2$ as the parametric representation:





Now we replace x by $x+a$ in the cartesian equation $(x-a)^2(x^2+y^2)=a^2y^2$ in order to translate the double point $(a,0)$ of the curve to the origin of the coordinate system. Next we intersect the curve with the line $y=t\cdot x$. We obtain one parametric representation for the whole strophoid by calculating the coordinates (x,y) of the point of intersection as a function of the parameter *t*. Check this with a graph for $a=2$:





5. Conclusions

Thanks to computer algebra systems it has become possible to eliminate parameters directly, without the use of determinants. This enables the treatment of elimination at an earlier stage and there is the additional advantage that the meaning of the concept elimination is clarified.

The student can quickly draw a graph and see that the parametric and cartesian representations yield the same curve. The choice of the window and the parameter interval is very important to obtain a nice graph. One needs to take the necessary time to discuss this with students as this provides insight into the geometrical meaning of a parameter.

One can "only" manipulate formulas with "computer algebra" and a computer cannot "see" the meaning of a letter in a given equation. It cannot recognise whether a letter represents a constant, a variable or a parameter. A CAS cannot determine what is given or what is asked, or what the set of numbers we are working with is. These are essential aspects of algebra (fortunately?) reserved for the human brain.

CAS stimulates mathematical reasoning, forces us to reflect on manipulation of formulas and to state explicitly the conditions that are often implicitly assumed when we work with paper and pencil.