EXPLORATIONS

Chapter 5

Riemann Sums and the Fundamental Theorem of Calculus

In calculus you study two types of integrals: indefinite integrals and definite integrals. *Indefinite integrals* are used to find the antiderivative of a function. *Definite integrals* can be used to find the area bounded by a function and the *x*-axis. In the following examples, you will discover a remarkable connection between these two types of integrals called the Fundamental Theorem of Integral Calculus.

Example 1: The area under a parabola

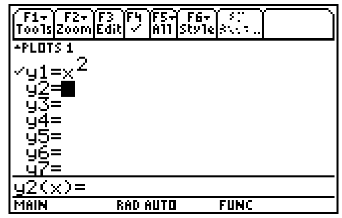
Example 1 begins with a numerical method to approximate the area bounded by a curve and then uses an analytic method that gives the exact area. The numerical method includes detailed graphs and rather lengthy calculations, which are performed with a calculator program. This program is listed at the end of this chapter. To duplicate the graphs and calculations, you will need to enter this program using the Program Editor in the APPS menu before beginning Example 1. If you don't want to enter the program, you can still study the numerical method and then use your TI-89 when you get to the analytic method.

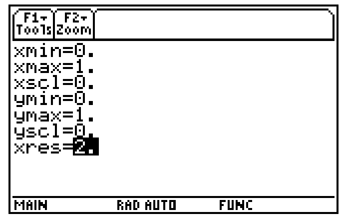
Find the area bounded by

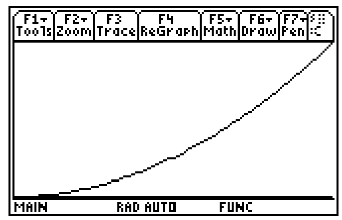
$$f(x) = x^2,\ x = 0,\ x = 1 \text{ and } y = 0.$$

Solution

Before finding the area using either the numerical or analytic method, graph the function. Enter the function as $y1$ in the Y= Editor. Enter the Window variable values shown here. Then graph the function.



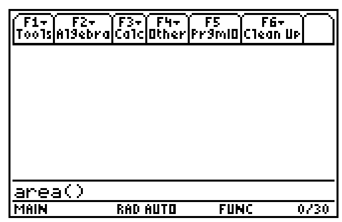




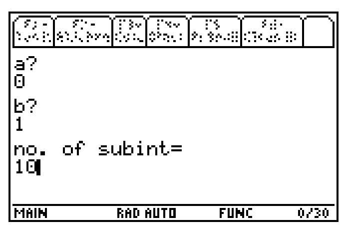
Numerical method

You can estimate the area under the curve with inscribed rectangles.

1. In the Program Editor, define the **area()** program listed at the end of this chapter.
2. After you have entered the function in $y1$ and graphed it, return to the Home screen, type **area(),** and press [ENTER] to run the program.

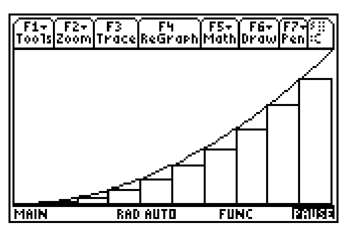


3. The program prompts the user for the left and right boundaries of the area and for the number of rectangles. Enter **0** and **1** for the boundaries and **10** for the number of rectangles.

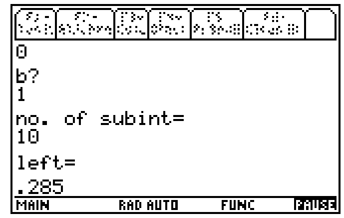


The program first inscribes rectangles with upper-left-hand corners on the curve.

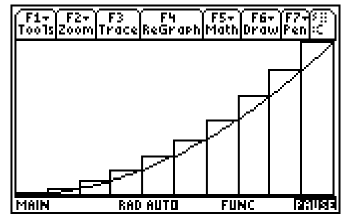
$f(x)=x^2$

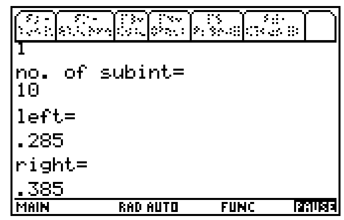


4. The sum of the areas of these left-hand rectangles approximates the area under the curve. This is called a left-hand rectangular approximation method, **lram**. Press [ENTER] to see the sum.



5. Next the program estimates the area with a right-hand rectangular approximation method, **rram.** The upper-right-hand corners of these rectangles lie on the curve. Press [ENTER] to see these rectangles, and then press [ENTER] again to see the sum of the areas of these rectangles.

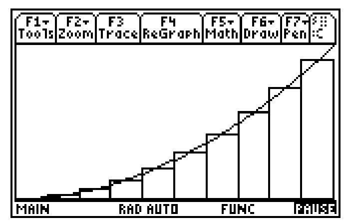


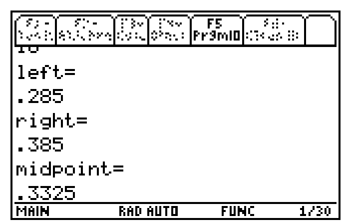


6. The program now uses rectangles with midpoints on the curve, called **mram**. Press [ENTER] to see these rectangles, and then press [ENTER] again to see the sum of their areas.

   The sums of the left, right, and midpoint rectangles are called *Riemann Sums*. Which of the three Riemann Sums do you think best approximates the area bounded by

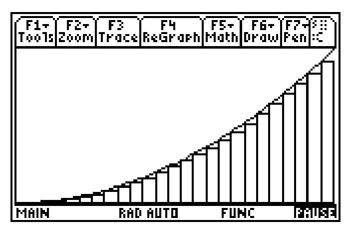
$$f(x) = x^2,\ x = 0, \text{ and } y = 0?$$



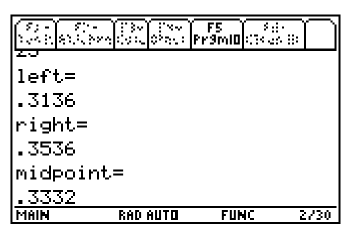


What do you think will happen if you increase the number of rectangles to 25?

Press [HOME] to return to the Home screen and run the **area()** program again with 25 subintervals.

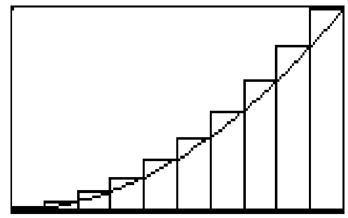


It appears that as the number of rectangles increases, they form a better fit to the area being estimated. Notice that the left, right, and midpoint estimates seem to be converging. From this, we estimate that the area is about .33.



Analytic method

You can find the exact value to which the left, right, and midpoint Riemann Sums converge with an analytic approach. First, define TI-89 functions for each of the rectangle methods **lram**, **rram,** and **mram**. Then take their limit as the number of rectangles approaches infinity in order to find the value to which they converge.



$a\ x_1\ x_2\ x_3\ x_4 \ldots \quad x_n = b$

Let's reexamine a picture of the right-hand rectangles in order to see how to define **rram** symbolically.

The area of each rectangle is given by the product of its height and width. If n is the number of rectangles and a and b are the left and right boundaries of the area, then the width of each rectangle is

$$h = \frac{b-a}{n}$$

The height of each rectangle is given by

$$f(x_1),\quad f(x_2),\quad f(x_3),\quad \ldots\quad f(x_n)$$

where

$$x_1 = a+h,\quad x_2 = a+2h,\quad x_3 = a+3h,\quad \ldots\quad x_k = a+k\cdot h$$

This leads to the following definition for **rram**:

$$rram = h\sum_{k=1}^{n} f(a+k\cdot h)$$

The definitions for **lram** and **mram** are similar in nature to **rram**. Study the TI-89 function definitions below, and try to understand how they work.

f(x)=x^2

lram*(a,b,n)=h*****Σ(f(a+h*k),k,0,n-1)|h=(b-a)/n*

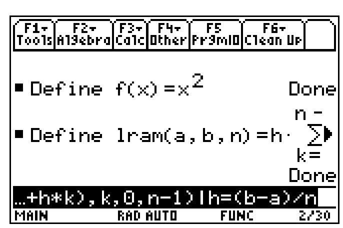
rram*(a,b,n)=h*****Σ(f(a+h*k),k,1,n)|h=(b-a)/n*

mram*(a,b,n)=h*****Σ(f(a+h*(.5+k)),k,0,n-1)|h=(b-a)/n*

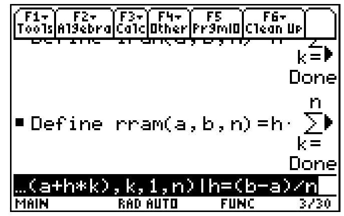
Define these functions on your TI-89 and find the limit to which they converge.

1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults.
2. Use the **Define** command to define the functions shown above. After you define **lram**, you can edit the entry line to change **lram** slightly to produce **rram** and **mram** so that you don't have to type their entire definitions.

   [CATALOG] **Define F** [(] **X** [)] [=] **X** [^] **2** [ENTER]

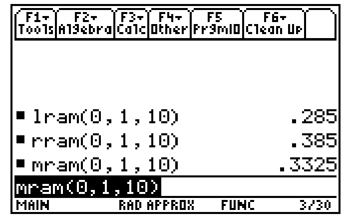


[CATALOG] **Define LRAM** [(] **A** [,] **B** [,] **N** [)] [=] **H** [×] [CATALOG] **Σ(F** [(] **A** [+] **H** [×] **K** [)] [,] **K** [,] **0** [,] **N** [−] **1** [)] [|] **H** [=] [(] **B** [−] **A** [)] [÷] **N** [ENTER]



3. Set the Exact/Approx mode setting on Page 2 of the MODE dialog box to **APPROXIMATE**.

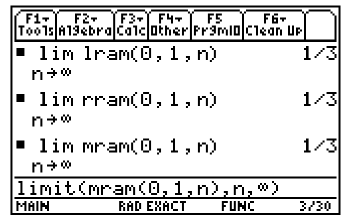
4. Check these functions by evaluating each one for $a = 0$, $b = 1$, and $n = 10$.



5. Set Exact/Approx= **EXACT**.

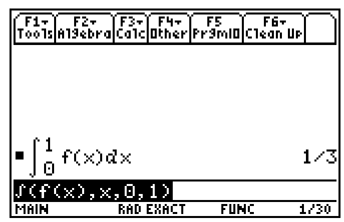
6. Take the limit of each summation function as the number of rectangles approaches infinity. The exact area is 1/3.

[CATALOG] **limit(LRAM** [(] **0** [,] **1** [,] **N** [)] [,] **N** [,] [◆] [∞] [)] [ENTER]

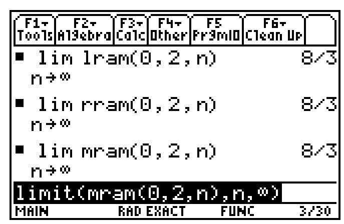


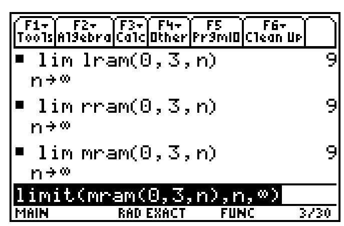
7. The definite integral $\int_0^1 f(x)dx$ also should equal the limit of the Riemann Sums.

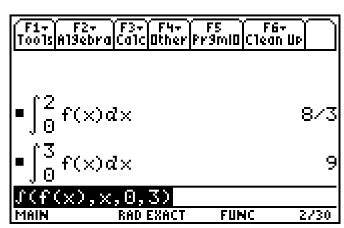
[2nd] [∫] **F** [(] **X** [)] [,] **X** [,] **0** [,] **1** [)] [ENTER]



8. Find the area for other values of b (b=2 and b=3). See if the definite integral gives the same result as the limits of the Riemann Sums.

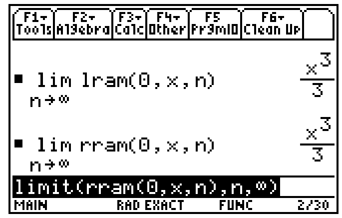




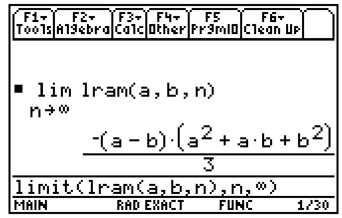


9. Predict the result for $a = 0$ and $b = x$. Verify your prediction on the TI-89.

   How is the area from $a = 0$ to $b = x$ related to *f(x)?* If you noticed the area function is the antiderivative of *f(x)*, you are on your way to discovering a relationship called the Fundamental Theorem of Calculus.

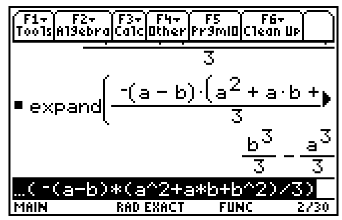


10. Use **lram** to find the area from a to b.



11. This answer is not in an easily recognizable form, so use the **expand(** command to expand the result. The result from step 10 is pasted from the history area to the entry line to save typing.

   [CATALOG] **expand(** ⏶ [ENTER] [ )] [ENTER]

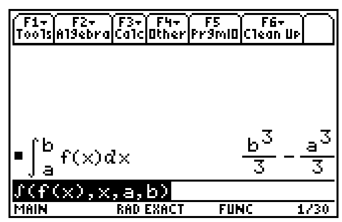


12. Predict $\int_a^b f(x)dx$.

   Verify your prediction on the TI-89.

   How is this answer related to $f(x)$?

   Did you notice there is an antiderivative of $f(x)$ involved?

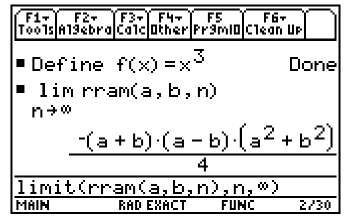


Example 2: The area under other curves

Redefine $f(x)$ and see if the relationship you observed in Example 1 works for other functions.

Solution

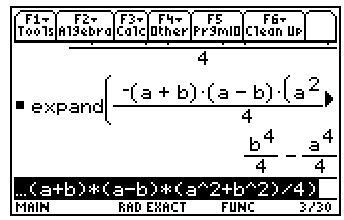
1. Press [2nd] [F6] **Clean Up** and select **2:NewProb** to clear variables and set other defaults.

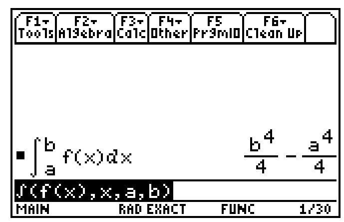


2. Use the **Define** command and define

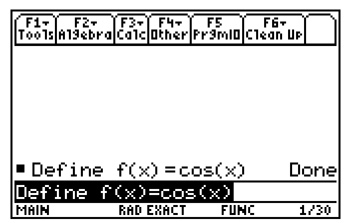
$$f(x) = x^3$$

3. Find the limit of the right-hand Riemann Sum for the area from a to b, and compare the result with the corresponding definite integral.

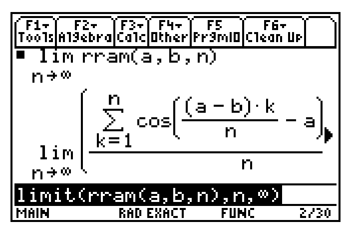


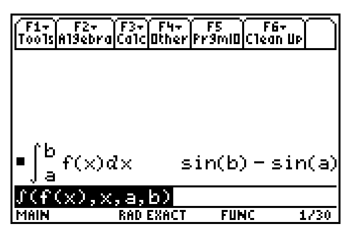


4. Define $f(x) = \cos(x)$.



5. Find the limit of the right-hand Riemann Sum and compare the result with the corresponding definite integral.





The TI-89 was not able to evaluate the limit of the Riemann Sums for $f(x)=\cos x$ since there is not a simple closed form expression for this sum; however, it was able to evaluate the definite integral. Even though the calculator could not evaluate the limit of the Riemann Sum for this function, the pattern from previous examples still holds. That is, the limit of the Riemann Sum is related to antiderivatives of $f(x)$.

Conclusion

In general, the limit of the Riemann Sums for the area bounded by a positive function $f(x)$, the vertical lines $x = a$ and $x = b$, and the x-axis is equal to $f(b) - f(a)$ where $f(x)$ is an antiderivative of $f(x)$. This result is called the Fundamental Theorem of Calculus.

Since this area is given by the definite integral

$$\int_a^b f(x)dx$$

the Fundamental Theorem also tells us that

$$\int_a^b f(x)dx = f(b) - f(a)$$

where $f(x)$ is an antiderivative of $f(x)$. This is a powerful shortcut.

Rectangular Area Approximation Program

Here is the program used to draw and evaluate the rectangular methods in this chapter. You must enter the function that bounds the area as $y1$ in the Y=Editor and enter the proper viewing Window variable values before running the program.

```
area()
Prgm
Prompt  a
Prompt  b
Input  "no. of subint=",n
(b-a)/n→h
h/2→d
Ø→l
Ø→m
Ø→r
ClrDraw
DispG
Ø→j
a→x
While  j<n
y1(x)+l→l
Line  x,Ø,x,y1(x)
Line  x,y1(x),x+h,y1(x)
Line  x+h,Ø,x+h,y1(x)
x+h→x
j+1→j
EndWhile
Pause
Disp  "left=",h*l*1.Ø
Pause
ClrDraw
DispG
Ø→j
a+h→x
While  j<n
y1(x)+r→r
Line  x-h,Ø,x-h,y1(x)
Line  x-h,y1(x),x,y1(x)
Line  x,Ø,x,y1(x)
x+h→x
j+1→j
EndWhile
Pause
Disp  "right=",h*r*1.Ø
Pause
ClrDraw
DispG
Ø→j
a+d→x
While  j<n
y1(x)+m→m
Line  x-d,Ø,x-d,y1(x)
Line  x-d,y1(x),x+d,y1(x)
Line  x+d,y1(x),x+d,Ø
x+h→x
j+1→j
EndWhile
Pause
h*m→m
Disp  "midpoint=",m*1.Ø
EndPrgm
```

Exercises

1. Evaluate **lram**, **rram,** and **mram** by hand for

   $f(x) = 2x^2 + 1$, a = 1, b = 2, and n = 4.

2. Use the TI-89 to evaluate the rectangular approximations in Exercise 1.
3. Use the TI-89 to take the limits of the rectangular methods from Exercise 1 as $n \to \infty$.
4. Use a definite integral to evaluate the limits of the rectangular methods from Exercise 1 as $n \to \infty$. Compute by hand and then confirm with the TI-89.
5. Evaluate **lram**, **rram**, and **mram** by hand for

   $f(x) = \cos(x)$, a = 0, $b = \pi$, and n = 4.

6. Use the TI-89 to evaluate the rectangular approximations in Exercise 5.
7. Use the TI-89 to take the limits of the rectangular methods from Exercise 5 as $n \to \infty$.

   Use a definite integral to evaluate the limits of the rectangular methods from Exercise 5 as $n \to \infty$. Compute by hand and then confirm with the TI-89.