Using the Document: EulersMethod.tns

On page 1.2, the derivative $y' = g(x, y)$ is defined in a Math Box. The default definition for y' is $g(x, y) = \frac{xy}{4\sqrt{1+x^2}}$. This expression can be changed by the user to allow for more in-depth and conceptual questions concerning Euler's Method. The initial condition, the endpoint x-value, and the number of Euler steps are also defined on page 1.2.

Page 1.3 is a Lists and Spreadsheet page that displays x_i, y_i, and $\Delta x \cdot g(x_i, y_i)$. Page 1.4 shows a graph of the points obtained using Euler's Method. The slider for n is used to change the number of steps and the slider for k is used to step through each Euler approximation

Suggested Applications and Extensions

Use the default initial value problem, $y' = \frac{xy}{4\sqrt{1+x^2}}$, $y(0) = 1$, to answer questions 1-3. The values for x_0, y_0, b, and n can be set either in a Math Box or by using a slider. The default values are $x_0 = 0$, $y_0 = 1$, $b = 6$, and $n = 6$. The numerical approximations are given on page 1.3, a Lists and Spreadsheet page, and a visualization of the approximation is given on page 1.4.

1. Use Euler's Method to approximate $y(6)$ for each of the following values for n: (i) $n = 6$, (ii) $n = 12$, (iii) $n = 24$. Which value of n do you think produces the best estimate for $y(6)$? Why?
2. Use Euler's Method to approximate $y(-3)$ for each of the following values for n: (i) $n = 6$, (ii) $n = 12$, (iii) $n = 24$. Which value of n do you think produces the best estimate for $y(-3)$? Why?
3. Use Euler's Method to approximate $y(6)$ for $n = 6$. Use separation of variables to find an expression for y in terms of x. Add the graph of $y = f(x)$ on page 1.4 and compare it to approximation produced by Euler's Method. Use the graph of $y = f(x)$ to explain why the Euler approximation for $y(6)$ is an underestimate of the true value for $y(6)$.

Additional Problems

1. Use Euler's Method with $n = 4$ to estimate $y(2)$ where $y(x)$ is the solution to the initial-value problem $y' = 3y - x$, $y(1) = 0$.
2. Use Euler's Method with $n = 8$ to estimate $y(2)$ where $y(x)$ is the solution to the initial-value problem $y' = xy^2 - \frac{1}{4}x^2$, $y(0) = 1$. Consider each step in this Euler approximation. Explain why the estimate for $y(2)$ is so much larger than the estimate for $y(1.75)$.
3. Use Euler's Method with $n = 8$ to estimate $y(4)$ where $y(x)$ is the solution to the initial-value problem $y' = x + y$, $y(0) = 1$. Find y'' in terms of x and y, and use this expression to explain why this approximation is an underestimate or an overestimate for the true value of $y(4)$.
4. Use Euler's Method with $n = 8$ to estimate $y(2)$ where $y(x)$ is the solution to the initial-value problem $y' = \frac{y}{1+x^2}$, $y(0) = -1$. Use separation of variables to find an expression for y in terms of x. Graph $y = f(x)$ and the Euler approximation on the same coordinate axes. Explain why the first

few Euler approximations are below the graph of $y = f(x)$ and the remaining approximations are above the graph of $y = f(x)$.

5. Use Euler's Method with $n = 8$ to estimate $y(\pi)$ where $y(x)$ is the solution to the initial-value problem $y' = \sin(x + y)$, $y(0) = 0$. Use $n = 16$ to estimate $y(\pi)$. Which estimate do you think is better? Why?
6. Use Euler's Method with $n = 8$ to estimate $y(-2)$ where $y(x)$ is the solution to the initial-value problem $y' = -x^2 y$, $y(0) = 1$. Use separation of variables to find an expression for y in terms of x. Graph $y = f(x)$ and the Euler approximation on the same coordinate axes. Find y'' and use this to explain why the Euler approximation for $y(-2)$ is an underestimate of the true value for $y(-2)$.
7. Let the function $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = y - x^2$ such that $f(0) = 1$.
   (a) The function f has a critical point at $x = 1.67835$. What is the y-coordinate of this critical point?
   (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Use $\frac{d^2y}{dx^2}$ to determine whether the critical point found in part (a) is a relative minimum, relative maximum, or neither. Justify your answer.
   (c) The function f has an inflection point at $x = \ln 2$. Use Euler's Method with $n = 10$ to estimate $y(1.67835)$ where $y(1) = 5 - e$. Is this approximation an overestimate or an underestimate. Justify your answer.