Looking at Polynomials as
Dynamic Mathematical Objects

Carl Leinbach
Gettysburg College
Getysburg, PA 17325

Introduction

In this workshop we will consider the problem of constructing the expression for a polynomial curve based on the dynamic properties of the curve. By that we mean a description of the behavior of the curve at certain points within the domain of the curve. We use TI-Interactive to do the algebraic manipulations and we concentrate on the hard work of solving the problem. We begin with a fanciful story to spark our students' interest.

> *You and your friend have just inherited a motor racing track. It is three miles long with one mile long straight a-ways and has been used for racing automobiles. The two of you decide that it would be much more exciting (and profitable) to make a moto-cross track (a track for racing motor cycles geared for off road travel) that might include a section having a jump. You also believe that the oval shape of the track is "boring" and decide to change the shape of the east end of the track to cut across the existing infield and include the jump in the middle. You are asked to design the new track*

Let's look at the skills that we will need for this project:
    To draw the existing track:
        1. Drawing parametric plots for two straight sections and two semi circles on either end of the straight sections.
        2. Since the plot feature of TI-Interactive, like that of the TI-89 and TI-92 calculators only accepts one range of t values for each plot screen, it will be necessary to translate our given range into the desired range for the plots. For example, for the semi circle on the west side of the track we want a range starting at $\pi/2$ and ending at $3\pi/2$. For the semicircle at the east end we will want the range to go from $-\pi/2$ to $\pi/2$. For the straight lines the range will go from -1 to 1.
    To construct the new section of track
        1. Determine an expression that meets certain predefined characteristics.
        2. Solve a system of linear equations
        3. Find the derivative (gradient of a function).
        4. Convert a function expression into a parametric expression.

Drawing the Existing Track

We begin at the left side of the track (west end) with the semi circle. Thus our range for t will

be $\pi/2 \le t \le 3\pi/2$. We set this by choosing from the Insert option on the main menu bar and choosing "Graph" and then "Parametric". On the graphing pallet that appears choose the "Format" button and set `tmin` to `pi/2` and `tmax` to `3pi/2`. You will also want to set the range for *x* and *y* as well as the grids, and axes. Inset a graph here and explore the options. You can remove it when you are finished with your exploration and want to leave this document in its original condition.

We begin by drawing the semi circle on the left (west end) of the track. Since the track is three miles long and the straight sections are one mile each, that means the circular sections are each 1/2 mile long. The circumference of the entire circle is 1 mile. Thus, the radius of the circle is: $1/(2\pi)$. Of course, you can do this in your head, but let's try out TI-Interactive. Click on the icon with the square root symbol pointing to a page. Notice that a calculator appears and also an equation box that looks like an equation editor box. In that box type: **solve(πr=1, r)**. This is the same command as on the TI-89 or 92. When you press the enter key, by default you see the solution of the equation below your command. Note that this was an algebraic solution. TI-Interactive will execute the same symbolic commands as the TI-89/92. You can remove the results of the trial exercise by placing your cursor after the equation box and pressing the back space key on your computer key board. The parametric equations of a circle with center at (c1,c2) and radius, r, are

$$x1(t):= c1 + r*\cos(t)$$
$$y1(t):= c2 + r*\sin(t)$$

If we were working in TI-Interactive, we could click on the graph below and see a graph box that will contain all of the information that we used to create the graph.
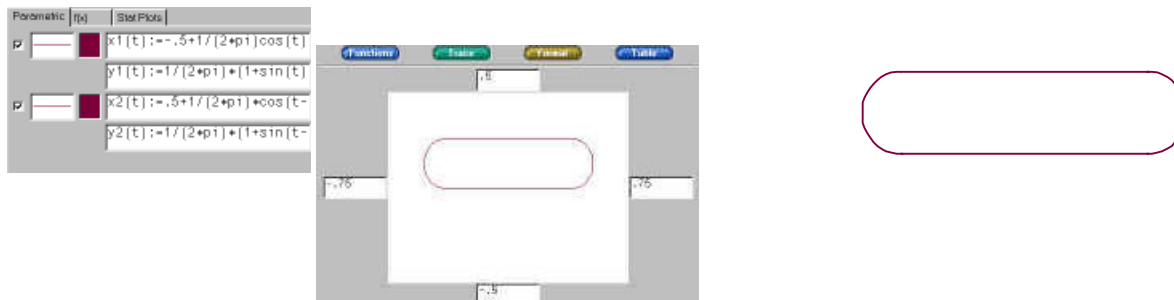
Note that we have chosen the parametric plot option and that the box appears the same as the Y= editor on the TI calculators. Also, since, like the calculators, the range of the parameter, t, must be the same for all graphs. We choose a range for t using the format button that appears above the graph. This means that we must do some translations to adjust the parameter to plot the part of the circles that we wish to display.

To graph the other half of the circle at the east end we merely change the position of the center and replace t by t - π. To find the equations for the straight sections, the y-coordinate is , of course, a constant, the x coordinate is found by transforming the interval $[\pi/2, 3\pi/2]$ into [-1,1]. We can find the transformation using TI-Interactive and the **solve** command

$$\text{solve}\left(x\cdot\left(\frac{\pi}{2}\right)+y=-1 \text{ and } x\cdot\left(\frac{3\cdot\pi}{2}\right)+y=1, \{x, y\}\right)$$

$$x=\frac{2}{\pi} \text{ and } y=-2$$

We leave it to you to check the parametric equations for the top straight section of the track.

Changing the Design of the Track

You decide to leave the top straight section the same and to cut a quarter of a mile off of the bottom one. You also realize that you must have a way to lead the traffic onto the top section. To do this last task, you decide to reduce the size of the semi-circle joining the to circle to one quarter of a mile in length. Use TI-Interactive to find the circumference of that section and the parametric equations of the section. You are going to have your other new section of track join this smaller semicircle at an angle of $-\pi/4$ with the horizontal.

By clicking on the graph, you can compare your answers with the second and fourth set of equations.
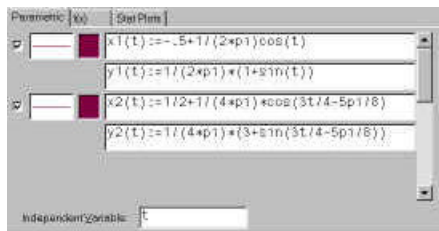
We are now ready for the main portion of this exercise, to find the expression for the section of track that cuts across the infield.

Finding a Polynomial to Meet Certain Criteria

At this point we know two things about the section of the track that is to be constructed. It leaves to bottom straight section at the point (1/4, 0) and meets the (new) small circle at the point where the radius makes an angle of $-\pi/4$ with the horizontal. Where is that point? Use TI-Interactive and list the values of (x2(t), y2(t)) when $t = \pi/2$. We can make these values for decimals by using the **approx(** command, but for now we leave the calcuations as exact.

$$\left[x2\!\left(\frac{\pi}{2}\right) \quad y2\!\left(\frac{\pi}{2}\right) \right]$$

$$\left[\frac{\sqrt{2}}{8\cdot\pi} + \frac{1}{2} \quad \frac{-\left(\sqrt{2}-6\right)}{8\cdot\pi} \right]$$

With only two pieces of information all we can do is draw a straight line joining the two points. This will not be a good solution since the line may join the circle along the tangent, but it will not make a smooth connection with the bottom. Try it with just a straight edge. The angle at the bottom will be very sharp. We want the places where the curves join to be 'smooth.' That is we want the entire track to look like one smooth closed curve. On the old track we did this without even thinking about it. We had the circles join the straight sections at the top and bottom, i.e. along the tangents to the semicircles We will do the same thing here. We will have the new stretch of track meet the old so that the tangents to the curve and the existing track are equal. That means that the first derivatives, or gradients, of the new section of track at these points must be the same as the gradients at the existing sections at the points where they join. We now have four pieces of information. That means we have enough equations to solve for four parameters.

It is a well known fact that a polynomial of degree n - 1 has n parameters. Thus, we are looking for a cubic polynomial.

$$f(x) := ax^3 + bx^2 + cx + d$$

We also need to know how to find an equation for the tangent to this polynomial at different values for x. Does this mean we need to teach calculus? Not really. We need to know how to find the linear approximation for a polynomial at a given point, $x=X$, i.e. given $f(x)$ we need to find constants, A, B, C, and D, such that

$$f(x) := A(x-X)^3 + B(x-X)^2 + C(x-X) + D .$$

Once we have found these constants, we use the fact that if x is "close to" X, the square term and cubic term become small thus the line

$$C(x-X) + D$$

is tangent to the graph of $f(x)$ at $x = X$.

We will use the algebraic capabilities of TI-Interactive to find A, B, C, and D. If we let $x - X = s$, then $x=X + s$.

$$f(x) := a\cdot x^3 + b\cdot x^2 + c\cdot x + d$$

"Done"

$$f(X+s)$$

$$a\cdot s^3 + \left(3\cdot a\cdot X + b\right)\cdot s^2 + \left(3\cdot a\cdot X^2 + 2\cdot b\cdot X + c\right)\cdot s + a\cdot X^3 + b\cdot X^2 + c\cdot X + d$$

$$f(x) := 3\cdot x^3 - 5\cdot x^2 + 2\cdot x + 1$$

"Done"

$$f(X+s)$$

$$3\cdot s^3 + \left(9\cdot X - 5\right)\cdot s^2 + \left(9\cdot X^2 - 10\cdot X + 2\right)\cdot s + 3\cdot X^3 - 5\cdot X^2 + 2\cdot X + 1$$

$$f(X + s) \,|\, X = 2$$

$$3 \cdot s^3 + 13 \cdot s^2 + 18 \cdot s + 9$$

Recall that $s = x - X$. or in the above case $s = x - 2$. Thus, the tangent line to the graph is

$$18(x - 2) + 9$$

Graphing this line and $f(x)$ we see

From the equation

$$f(X + s)$$

$$a \cdot s^3 + \left(3 \cdot a \cdot X + b\right) \cdot s^2 + \left(3 \cdot a \cdot X^2 + 2 \cdot b \cdot X + c\right) \cdot s + a \cdot X^3 + b \cdot X^2 + c \cdot X + d$$

we see that the slope of the tangent line to a general cubic at $x = X$ is

$$3aX^2 + 2bX + c$$

We will use this to help in solving for a, b, and c.

The equations that we have to work with to find the expression for the new section of track can be found using TI-Interactive

$$f(x) := a \cdot x^3 + b \cdot x^2 + c \cdot x + d$$

$$\text{"Done"}$$

$$f\left(\frac{1}{4}\right) = 0 \rightarrow eq1$$

$$\frac{a}{64} + \frac{b}{16} + \frac{c}{4} + d = 0$$

$$f\left(x2\left(\frac{\pi}{2}\right)\right) = y2\left(\frac{\pi}{2}\right) \rightarrow eq2$$

$$\frac{a \cdot \left(\sqrt{2} + 4 \cdot \pi\right)^3}{512 \cdot \pi^3} + \frac{b \cdot \left(\sqrt{2} + 4 \cdot \pi\right)^2}{64 \cdot \pi^2} + c \cdot \left(\frac{\sqrt{2}}{8 \cdot \pi} + \frac{1}{2}\right) + d = \frac{-\left(\sqrt{2} - 6\right)}{8 \cdot \pi}$$

The third and fourth equations are a result of the way in which the curve joins the existing portions of the new track. One of the equations is a result of smoothly meeting the bottom straight section of the track.

$$t(x) := 3 \cdot a \cdot x^2 + 2 \cdot b \cdot x + c$$

"Done"

$$t\left(\frac{1}{4}\right) = 0 \rightarrow eq3$$

$$\frac{3 \cdot a}{16} + \frac{b}{2} + c = 0$$

The other comes from meeting the small circle along a tangent that has slope 1.

$$t\left(\frac{\sqrt{2}}{8 \cdot \pi} + \frac{1}{2}\right) = 1 \rightarrow eq4$$

$$\frac{3 \cdot a \cdot \left(\sqrt{2} + 4 \cdot \pi\right)^2}{64 \cdot \pi^2} + \frac{b \cdot \left(\sqrt{2} + 4 \cdot \pi\right)}{4 \cdot \pi} + c = 1$$

Note how simply we were able to create the equations using the symbolic capabilities of TI-Interactive. We could take these equations and put the coefficients in a matrix and use the **rref(** command, but we will use the **solve(** command as above.

$$\text{approx}(\text{solve}(eq1 \text{ and } eq2 \text{ and } eq3 \text{ and } eq4, \{a, b, c, d\}))$$

$$a = -2.04172 \text{ and } b = 4.10182 \text{ and } c = -1.66808 \text{ and } d = .19256$$

For convenience, since we may at some later point change the conditions for the cubic expression, we will store the particular cubic that we have found as $g(x)$.

$$g(x) := -2.04172 \cdot x^3 + 4.10182 \cdot x^2 - 1.66808 \cdot x + .19256$$

"Done"

Now we are ready to draw the track.

On the calculator screen we would have can do this by using the Draw menu and specifying DrawFunc(when(x>1/4 and x<$\frac{\sqrt{2}}{8p} + \frac{1}{2}$,g(x)). In TI-Interactive, we need only copy the graph, click on the graph and choose the f(x) tab in the equation editor. This will draw a function y=f(x) with the existing parametric plots.
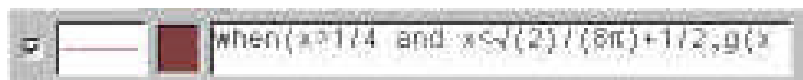
The competed shape of the track is shown in the figure at the top of the following page.

Exercises

1. Change the west end of the track in a manner similar to the way the east end was changed, but put the small circular section at the bottom of the track.

2. Suppose the curve that we constructed is not "exciting" enough. There is a possibility of a water or mud jump for the motor cycles if we guarantee the new section of track passes through the point (.4, .1).

   a. Verify that this point is not already on your section of track.

   b. Assuming that it is not, what degree polynomial do you now use to construct the segment of track?

   c. Construct that segement using the techniques shown above.