

Integration by Parts

ID: 9850

 Time required
 45 minutes

Activity Overview

In previous activities, students have explored the differential calculus through investigations of the methods of first principles, the product and quotient rules. In this activity the product rule becomes the basis for an integration method for more difficult integrals.

Topic: Techniques of Integration

- *Derive the formula for integration by parts and use it to compute integrals.*
- *Use successive applications of integration by parts to compute integrals.*
- *Use **Integral** (on the Calculus menu) to verify manual calculation of integrals.*

Teacher Preparation and Notes

- *This activity can serve to consolidate earlier work on the product rule and methods of integration. It offers a suitable introduction to the method of integration by parts.*
- *This activity is designed to be **teacher-led**. Have the students work in pairs. Use the following pages to present the material to the class and encourage discussion. Although the majority of the ideas and concepts are presented in **this** document, be sure to cover all the material necessary for students' total comprehension.*
- *The students will need to be able to compute the derivative and the indefinite integral on their own.*
- *Answers should be completed on the student worksheet.*
- *Before starting this activity, students should go to the home screen and select **F6:Clean Up > 2:NewProb**, then press **ENTER**. This will clear any stored variables, turn off any functions and plots, and clear the drawing and home screens.*
- ***To download the student worksheet, go to education.ti.com/exchange and enter "9850" in the quick search box.***

Associated Materials

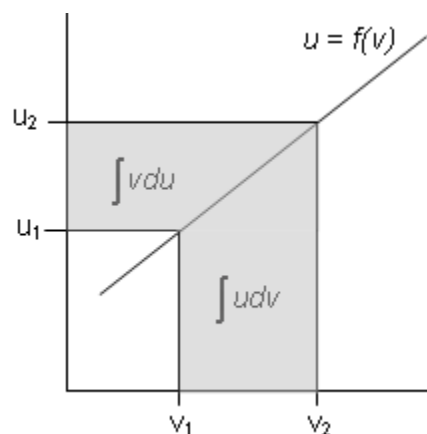
- *IntegrationByParts_Student.doc*

Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

- *Exploring the Area Under a Curve – (TI-Nspire Technology) — 8268*
- *Area Accumulation (TI-Nspire Technology) — 12241*
- *Integration by Substitution (TI-89 Titanium) — 9889*
- *Integration (TI-89 Titanium) — 5509*

- Step 1:** Begin with discussion concerning integration methods and review standard integrals. Some review of substitution methods of integration, such as $\int \cos(2x + 1)dx$, is recommended. Students should be aware that not all functions can be integrated symbolically.
- Step 2:** Students are to complete exercises 1 and 2 on the worksheet. They will use the product rule on the function $f(x) = \sin(x)\ln(x)$. Students should be given opportunities to review and consolidate their skills and understandings related to both product rule and the process of integration (particularly drawing distinction between the integral of a function and the area under the curve of the graph of that function).
- Step 3:** The idea of taking the integral of a derivative may well be new to students and should be handled carefully, ensuring that the errors and misunderstandings are exposed if possible. Students should be aware that integrals evaluated by CAS generally do not include the constant term.
- Step 4:** Students are to find the integral of both sides of the product rule. Integrating the product rule and rearranging the parts provides an expression which proves to be the basis for our method of integration by parts. However, it is recommended that this process is not assumed to be valid, but that students be given the opportunity to engage with it, to discuss and to challenge.
- Step 5:** Students are to complete exercise 6 on the worksheet. A graphical approach can help students appreciate the significance of each of the components parts of integration by parts statement: The areas under the curve **with respect to u** and **with respect to v** are key concepts here. Interestingly, it is the product $u \cdot v$ which is most difficult to draw from this scenario. It represents a shorthand for the difference product, $u_2 \cdot v_2 - u_1 \cdot v_1$. This is shown on the diagram as the L-shaped area between the curve and the axes, between the given limits.
- Step 6:** Drawing together these observations, students should be assisted to see the significance of this method. Integration by parts allows students to find the integral of a function that is a product by taking the derivative of one component and the integral of the other.



Step 7: Students are to complete exercise 7 by integrating $\ln(x)$ using integration by parts. They can use the home screen on the calculator to check their work.

The challenge for students when using integration by parts is which part to make u and which part to make dv . Whether this choice makes a difference is an extension exercise.

Even though this function does not appear to be a product we can treat it as one.

Step 8: Students are to complete the second part of finding the integral of $\sin(\ln(x))$. They will need to use integration by parts to find the integral of $\cos(\ln(x))$ and then substitute their result into the result for $\sin(\ln(x))$. The integral check is the screen to the right.

This exercise with its companion below shows the cyclical work for the trig functions.

Step 10: Students are to use integration by parts to find the integrals of the four functions on their worksheet.

Reduction rules may also be developed in the way 10b is worked. This method leads to a method for integrating trig functions of higher powers.

Step 11: The extension challenges are added in the student solutions.

F1+ Tools	F2+ Algebra	F3+ Calc	F4+ Other	F5 Pr3mID	F6+ Clean Up
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$$\int \ln(x) dx = x \cdot \ln(x) - x$$

$f(\ln(x), x)$

MAIN RAD AUTO FUNC 1/30

F1+ Tools	F2+ Algebra	F3+ Calc	F4+ Other	F5 Pr3mID	F6+ Clean Up
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$$\int \cos(\ln(x)) dx = \frac{x \cdot \cos(\ln(x))}{2} + \frac{x \cdot \sin(\ln(x))}{2}$$

$f(\cos(\ln(x)), x)$

MAIN RAD AUTO FUNC 1/30

F1+ Tools	F2+ Algebra	F3+ Calc	F4+ Other	F5 Pr3mID	F6+ Clean Up
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$$\int \sin(\ln(x)) dx = \frac{x \cdot \sin(\ln(x))}{2} - \frac{x \cdot \cos(\ln(x))}{2}$$

$f(\sin(\ln(x)), x)$

MAIN RAD AUTO FUNC 1/30

F1+ Tools	F2+ Algebra	F3+ Calc	F4+ Other	F5 Pr3mID	F6+ Clean Up
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$$\int \tan^2(x) dx = x \cdot \tan^2(x) - \frac{\ln(x^2 + 1)}{2}$$

$$\int (x^2 \cdot e^x) dx = (x^2 - 2 \cdot x + 2) \cdot e^x$$

$f(x^2 \cdot e^x, x)$

MAIN RAD AUTO FUNC 2/30

Solutions

1. $\frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$
2. $\frac{d}{dx}(\sin(x) \cdot \ln(x)) = \sin(x) \cdot \frac{1}{x} + \ln(x) \cdot \cos(x)$
3. No; While $\frac{d}{dx}(\int f(x) dx) = f(x)$ is true, $\int \frac{d}{dx}(f(x)) dx = f(x) + C$
4. $\int \left(\frac{d}{dx}(u(x) \cdot v(x)) \right) dx = u(x) \cdot v(x) + C$
5. $v(x) \cdot \int u(x) dx + u(x) \cdot \int v(x) dx$
6. The area between the curve $u = f(v)$ and the v -axis (between the specified limits, i.e. $\int_{v_1}^{v_2} u \cdot dv$) can be found by taking the area of the large rectangle ($u_2 \cdot v_2$) minus the smaller rectangle ($u_1 \cdot v_1$), and then subtracting the area of the region between the curve and u -axis, between the specified limits ($\int_{u_1}^{u_2} v \cdot du$).
7. $\int (\ln(x) \cdot 1 dx)$ using Integration by Parts

$$u(x) = \ln(x) \quad du = \frac{1}{x} dx$$

$$dv = 1 dx \quad v = x$$

$$\int \ln(x) \cdot 1 dx = \ln(x) \cdot x - \int x \cdot \frac{1}{x} dx = x \cdot \ln(x) - x$$
8. $\int \cos(\ln(x)) dx = x \cdot \cos(\ln(x)) + \int \sin(\ln(x)) dx$
9. $\int \sin(\ln(x)) \cdot 1 dx = x \sin(\ln(x)) - 2 \left[x \cos(\ln(x)) + \int \sin(\ln(x)) dx \right]$

$$= \frac{x}{2} \left[\sin(\ln(x)) - \cos(\ln(x)) \right]$$
10. a. $\int \tan^{-1}(x) \cdot 1 dx$ using Integration by Parts

$$u(x) = \tan^{-1}(x) \quad du = \frac{1}{x^2 + 1} dx$$

$$dv = 1 dx \quad v(x) = x$$

$$\int \tan^{-1}(x) \cdot 1 dx = \tan^{-1}(x) \cdot x - \int x \cdot \frac{1}{x^2 + 1} dx$$

$$= x \tan^{-1}(x) - \frac{1}{2} \ln(x^2 + 1) + C$$

b. $\int x^2 \cdot e^x dx$ using Integration by Parts

$$u(x) = x^2 \quad du = 2x$$

$$dv = e^x dx \quad v(x) = e^x$$

$$\int x^2 \cdot e^x dx = x^2 \cdot e^x - \int e^x \cdot 2x dx$$

Repeat Integration by Parts for $\int e^x \cdot 2x dx$

$$u(x) = 2x \quad du = 2 dx$$

$$dv = e^x dx \quad v(x) = e^x$$

$$\int e^x \cdot 2x dx = 2x \cdot e^x - 2 \int e^x dx = 2x \cdot e^x - 2e^x$$

Hence, $\int x^2 \cdot e^x dx = x^2 \cdot e^x - 2x \cdot e^x + 2e^x = e^x(x^2 - 2x + 2)$

c. $\int x \tan^{-1}(x) dx$ using Integration by Parts

$$u(x) = \tan^{-1}(x) \quad du = \frac{1}{x^2 + 1} dx$$

$$dv = x dx \quad v(x) = \frac{x^2}{2}$$

$$\begin{aligned} \int x \tan^{-1}(x) dx &= \tan^{-1}(x) \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x^2 + 1} dx \\ &= \frac{x^2 \tan^{-1}(x)}{2} - \frac{1}{2} \int \frac{x^2 + 1}{x^2 + 1} - \frac{1}{x^2 + 1} dx \\ &= \frac{x^2 \tan^{-1}(x)}{2} - \frac{1}{2} (x - \tan^{-1}(x)) \\ &= \frac{1}{2} ((x^2 + 1) \tan^{-1}(x) - x) \end{aligned}$$

d. $\int x \cos(2x + 1) dx$ using Integration by Parts

$$u(x) = x \quad du = 1 dx$$

$$dv = \cos(2x + 1) dx \quad v(x) = \frac{\sin(2x + 1)}{2}$$

$$\begin{aligned} \int x \cos(2x + 1) dx &= x \cdot \frac{\sin(2x + 1)}{2} - \int \frac{\sin(2x + 1)}{2} \cdot 1 dx \\ &= \frac{x \sin(2x + 1)}{2} + \frac{\cos(2x + 1)}{4} \\ &= \frac{1}{4} (2x \sin(2x + 1) + \cos(2x + 1)) \end{aligned}$$

11. Often it does not matter, if both functions are readily differentiable and integrable, but in cases where one may not be reduced by the process, it can make a difference.

12. No, since the quotient rule formula does not allow us to easily rearrange the parts—the denominator v^2 causes problems.