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**Leonardo da Pisa
(Fibonacci)
c. 1180 - 1250**

Leonardo da Pisa, considered by some scholars to have been the greatest mathematician of the Middle Ages, published his famous treatise, *Liber abaci* (book of the abacus) in 1202. In this book, Leonardo da Pisa, better known as Fibonacci, presented the following problem:

How many pairs of rabbits will be produced in a year, beginning with a single pair if in every month each pair bears a new pair which becomes productive exactly two months after birth?

By drawing a tree diagram showing the number of pairs of rabbits at the end

of each month, we discover that the number of pairs of rabbits in the first six months are given by the following sequence, 1, 1, 2, 3, 5, 8, ... This famous sequence, named the *Fibonacci* sequence in honor of Leonardo da Pisa has the property that each term is the sum of the two previous terms.

In the tree diagram on the right, we mark the first pair of rabbits with the number 1. Since they do not produce offspring in the first 2 months, there is only one pair at the end of two months. On the first day of the third month, they give birth to another pair (labelled 2) so there are two pairs at the end of the third month. The arrows in the tree diagram show the offspring in the successive months up to the end of the sixth month.

The Fibonacci sequence is one of an infinite number of possible sequences. To define a sequence, we merely assign to each positive integer n a particular number $u(n)$. The number $u(1)$ is taken to be the first term of the sequence, $u(2)$ is the second term of the sequence, ... and $u(n)$ is the n^{th} term. If we define $u(n)$ as an algebraic expression, then we have defined all the terms of the sequence. In effect, a sequence is a function defined on the positive integers (rather than on the real numbers). In general,

Definition: Any function u defined for all positive integers from 1 to N is said to define a *sequence of N terms*, and $u(n)$ is said to be the n^{th} term of the sequence u .

If a sequence has an infinite number of terms, we call it an *infinite sequence*.

Arithmetic Sequences

The first five terms of the sequence $u(n) = 5 + 3(n - 1)$ are: 5, 8, 11, 14, and 17. Each term is 3 greater than the previous term. That is, $u(n) - u(n-1) = 3$. Such a sequence in which successive terms differ by a constant is called an *arithmetic sequence*. This constant is called the *common difference*.

The first few terms of a general arithmetic sequence with first term a and common difference d are given by:

$$a, a + d, a + 2d, \dots, \boxed{a + (n - 1)d}, \dots$$

\uparrow
common difference d
 \uparrow
 n^{th} term $u(n)$

\leftrightarrow newborn pair

End of Month	The Fibonacci Rabbit Problem	Number of Pairs
1		1
2		1
3		2
4		3
5		5
6		8

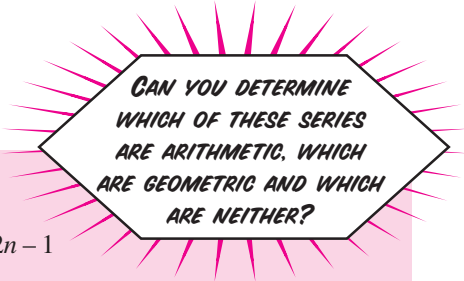
WORKED EXAMPLES

Geometric Sequences

The first five terms of the sequence defined by $u(n) = 2^n$ are: 2, 4, 8, 16 and 32. Each term in the sequence is double the previous term. That is, $u(n) = 2u(n-1)$. Such a sequence in which successive terms are in a fixed ratio is called a *geometric sequence*. The fixed ratio is called the *common ratio*. The first few terms of a general geometric sequence with first term, a and common ratio, r are given by:

$$a, ar, ar^2, \dots, ar^{n-1}, \dots$$

↑ common ratio r
↑ n^{th} term $u(n)$



WORKED EXAMPLE 1

Write the first 10 terms of each sequence and evaluate its 20th term.

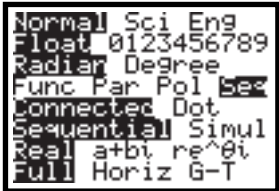
- a) $u(n) = 4 - 7(n - 1)$ b) $u(n) = 2^n$ c) $u(n) = \left(\frac{3}{4}\right)^n$ d) $u(n) = 3n^2 + 2n - 1$

SOLUTION

Method 1

a) We set the calculator in **sequence** mode by pressing:

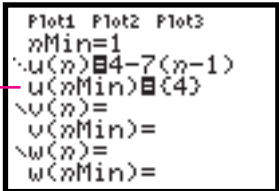
```
MODE [v] [v] [v] [▶] [▶] [▶] ENTER
```



Step 1: Define the sequence.

To define the sequence $u(n) = 4 - 7(n - 1)$, we press:

```
Y= [4] [-] [7] [(] [X,T,θ,n] [-] [1] [)] ENTER
```



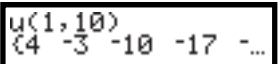
Step 2: Set the first term by assigning a value to $u(nMin)$.

When $n = 1$, $u(n) = 4$, so we set $u(nMin) = \{4\}$ by pressing **4** **ENTER**

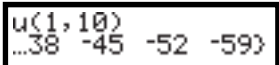
Step 3: Evaluate $u(1,10)$ and $u(20)$.

Return to the home screen by pressing: **2nd** [QUIT]. To enter $u(1, 10)$ press:

```
2nd [u] [(] [1] [,] [1] [0] [)] ENTER
```



This yields the first four terms of the sequence as shown in the display. By pressing the **▶** key repeatedly, we can scroll to the right and obtain the next 6 terms of the sequence: -24, -31, -38, -45, -52, and -59. To evaluate the 20th term, we press:



2nd [u] [(] [2] [0] [)] **ENTER** to obtain $u(20) = -129$.



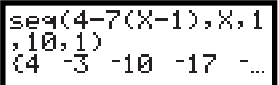
Method 2

Alternatively, we can generate the sequence $u(n)$ in function mode by using the **seq**(command on the LIST menu. To access this command, we press: **2nd** [LIST] **▶** **5**

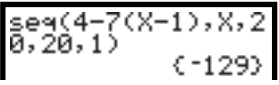
We then enter the five arguments as shown.

seq(**4 - 7(x - 1)**, **X**, **1**, **10**, **1**) **ENTER** yields the first 10 terms

↑ sequence
 ↑ variable
 ↑ initial value
 ↑ final value
 ↑ increment



seq(**4 - 7(x - 1)**, **X**, **20**, **20**, **1**) **ENTER** yields the 20th term -129.



- b) Proceeding as in Part a), we obtain {2, 4, 8, 16, 32, 64, 128, 256, 512, 1024} and $u(20) = 1\,048\,576$.
 c) As in Part a), we obtain { .750, .563, .422, .316, .237, .178, .133, .100, .075, .056 } and $u(20) = .003$.
 d) As in Part a), we obtain {4, 15, 32, 55, 84, 119, 160, 207, 260, 319} and $u(20) = 1239$.

WORKED EXAMPLES

We observed that the Fibonacci sequence was defined by relating the n^{th} term $u(n)$ to the two preceding terms, $u(n-1)$ and $u(n-2)$. Also we saw how the arithmetic and geometric sequences could be defined by relating the n^{th} term $u(n)$ to the preceding term $u(n-1)$. Sequences which are defined by relating the n^{th} term to one or more previous terms are said to be defined *by recursion*. Such sequences are very important in modeling dynamic systems such as population growth, where the population of the n^{th} generation is a function of the population of the $(n-1)^{\text{th}}$ generation.



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WORKED EXAMPLE 2

Graph the first 10 terms of each sequence and evaluate the 20th term.

- | | |
|---|--|
| <p>a) $u(n) = u(n-1) - 7$
$u(0) = 4$</p> | <p>b) $u(n) = 2u(n-1)$
$u(0) = 2$</p> |
| <p>c) $u(n) = u(n-1) + 2\sqrt{u(n-1)} + 1$
$u(0) = 1$</p> | <p>d) $u(n) = \frac{1}{u(n-1)} + 1$
$u(0) = 1$</p> |

SOLUTION

a) Before defining each sequence, we press:

MODE **▼** **▼** **▼** **▶** **▶** **▶** **ENTER**

to obtain SEQUENCE mode. Then we press:

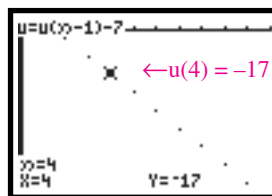
Y= **2nd** **[u]** **(** **X,T,θ,n** **-** **1** **)** **-** **7** **ENTER**

To define the first term $u(1) = 4$, we press: **4** **ENTER** and the brackets { } are placed around the 4 as shown in the display.

To graph the first 10 terms of $u(n)$, we press **WINDOW** and set the window variables as shown.

When we press **GRAPH** we obtain the graph shown in the display.

When we press **TRACE** and then repeatedly press the **▶** key, we can move along the graph of the sequence as shown. The display shows that $u(4) = -17$.



```
Plot1 Plot2 Plot3
nMin=1
u(n)Bu(n-1)-7
u(nMin)B(4)
```

```
WINDOW
nMin=1
nMax=10
PlotStart=1
PlotStep=1
Xmin=0
Xmax=11
Xscl=1
Ymin=-60
Ymax=4
Yscl=1
```

n	u(n)
14	-87
15	-94
16	-101
17	-108
18	-115
19	-122
20	-129

n=20

```
TABLE SETUP
TblStart=20
ΔTbl=1
Indent: Auto Ask
Depend: Auto Ask
```

To obtain the 20th term, we could reset the window variables to include $n = 20$. Alternatively, we can press **2nd** **[TABLE]** to obtain a table of values, and then scroll down to $n = 20$.

We observe that $u(20) = -129$, as we found in *Worked Example 1*.

To begin the table of values at $n = 20$, we can press **2nd** **[TBLSET]** to obtain the table setup menu shown in the display. We then enter 20 opposite TblStart and press **2nd** **[TABLE]**.

- b) Proceeding as in Part a) yields the same sequence as in Part b) of *Worked Example 1*.
 c) Proceeding as in Part a), we obtain the sequence of squares $u(n) = n^2$ and $u(20) = 400$.
 d) Proceeding as in Part a), we discover that the terms of the sequence approach 1.618..., the *golden ratio*...hmmm.

EXERCISES & INVESTIGATIONS

1. Write a sentence to define each of the following terms. Give an example of each.
- a finite sequence
 - an infinite sequence
 - an arithmetic sequence
 - a geometric sequence

2. Indicate whether the following statement is true or false:
An infinite sequence is a function defined on the positive integers.
 Justify your answer.

3. List the first 10 terms of each sequence.

- $u(n) = 95 - 9(n-1)$
- $u(n) = 3^{n-1}$
- $u(n) = 8 - 2^n$
- $u(n) = 2^{-n}$
- $u(n) = \left(1 + \frac{2}{n}\right)^n$

Identify each sequence as an arithmetic sequence, geometric sequence, or neither type.

4. Calculate the 18th term of each sequence in exercise 4.

5. a) Graph the first 20 terms of sequences $u(n) = 5 + 2^{-n}$ and $u(n) = \left(1 + \frac{2}{n}\right)^n$ in window: $0 \leq x \leq 21$ and $-1 \leq y \leq 10$.
- b) Trace along each graph in a) to evaluate the 20th term of each sequence.
- c) Create a table to check your answer in Part b). Which answer do you think is more accurate? Explain.

6. List the first 15 terms of each sequence.

- $u(n) = \sqrt{u(n-1)}$ first term: 2
- $u(n) = u(n-1) - u(n-1)^2$ first term: 0.75
- $u(n) = \cos u(n-1)$ first term: 0
- $u(n) = 2.8u(n-1)[1 - u(n-1)]$ first term: 0.6

7. Compare the sequences in *Worked Examples 1 and 2*. Find a pair of sequences which are the same. Explain why the two different definitions generate the same sequence. Can every arithmetic sequence be written in both function and recursive form? Is this true for every geometric series?

8. Draw the next row in the Fibonacci rabbit tree diagram on page 8 to verify that there are 13 rabbits at the end of the seventh month.

9. Define the *Fibonacci sequence* using the definition of $u(n)$ shown and create a table to show its first 20 terms. What value does $u(n+1) / u(n)$ approach as n becomes large?

```

Plot1 Plot2 Plot3
nMin=1
u(n)=u(n-1)+u(n-2)
u(nMin) (1,1)
    
```

10. In 1845, P. F. Verhulst formulated a law for population growth expressed by the *logistic equation*,

$$u(n) = \lambda u(n-1)[1 - u(n-1)]$$

where $u(n)$ denotes the population of the n^{th} generation and λ is a constant associated with a particular species.

- a) Assuming an initial population of $u(0) = 0.6$ units, graph the populations of the first 40 generations (i.e. $u(n)$ vs. n) for each of these values of λ :

- (i) $\lambda = 1.6$ (ii) $\lambda = 3.1$ (iii) $\lambda = 3.5$

Hint: Set the window variables as shown in this display by pressing **WINDOW**

```

WINDOW
nMin=1
nMax=40
PlotStart=1
PlotStep=1
Xmin=-1
Xmax=45
Xscl=1
Ymin=-.2
Ymax=1
Yscl=1
    
```

For each value of λ , trace along your graph to find the population of the 40th generation, i.e. $u(40)$.

- b) For each value of λ , indicate whether there is a limiting “stable value” toward which the population converges. That is, for each value of λ , indicate whether there is some limiting population L that $u(n)$ approaches as n gets large. Justify your answer by interpreting your graphs.

- c) To compare the n^{th} term of a sequence with the $n-1^{\text{th}}$ term for large n , mathematicians have developed the so-called *web plot* (because it resembles a spider web). The web plot is a scatterplot of the points $(u(n-1), u(n))$.

If the sequence $u(n)$ approaches a limit for large n , then the points in the web plot will lie on or close to the line $y = x$. Any point $(u(n-1), u(n))$ on a web plot projects horizontally onto the point $(u(n), u(n))$ on the line $y = x$. Why? The point $(u(n), u(n))$ on the line $y = x$, projects vertically onto the point $(u(n), u(n+1))$ on the web plot. Why? Therefore, starting at any point on the web plot, with vertical coordinate $u(n)$ and projecting horizontally onto $y = x$ and then vertically onto the web plot, maps us onto a point with vertical coordinate $u(n+1)$. This provides a visual map of the long term behavior of $u(n)$.

To create web plots of the three logistic sequences above, press **2nd** [**FORMAT**] **▶** **ENTER** . Then graph the sequences using these window settings and trace.

For $\lambda = 1.6$

```

WINDOW
nMin=1
nMax=100
PlotStart=1
PlotStep=1
Xmin=.35
Xmax=.4
Xscl=1
Ymin=.35
Ymax=.4
Yscl=1
    
```

For $\lambda = 3.1$ and $\lambda = 3.5$

```

WINDOW
nMin=1
nMax=100
PlotStart=1
PlotStep=1
Xmin=0
Xmax=1
Xscl=1
Ymin=0
Ymax=1
Yscl=1
    
```

Write a brief report to describe and interpret your web plots.