NUMB3RS Activity: Critical Maths
Episode: “End of Watch”

Topic: Critical Path Analysis (CPA) **Grade Level:** 9 - 12
Objective: To understand the mathematics of CPA for scheduling and completing a complicated project
Time: 25 - 30 minutes

Introduction

In “End of Watch,” Charlie is trying to understand the possible paths taken by a murder suspect. He initially considers using Ant Colony Optimization (featured in a separate activity for this episode). He later realizes that he was only thinking of the paths of the suspect, and not the paths of the victim. He refers to and discusses Critical Path Analysis (CPA) as a mathematical way of finding the most efficient way to complete a set of tasks. In the case of the victim, it was the timeline for the day he was killed.

Discuss with Students

In the 1950s, NASA and the US military developed CPA, originally called PERT (Project Evaluation and Review Tables), to support the fledgling US space program. Hundreds of companies had to produce thousands of parts just in time to ensure that the first rockets were completed by their launch date.

CPA can be used for many different complex projects that can be broken down into a list of tasks, assuming it is known how long it takes to complete each task, and which tasks must come before others. For example, when building a house, it would not be a good idea if the carpenters showed up to build the walls before the foundation was finished. Other examples of projects that can use CPA are producing a school yearbook, opening a new restaurant, building a space shuttle, designing and marketing a new product, or in the case of Charlie’s example in this episode, preparing a holiday meal.

According to Charlie, “You need to prepare and cook each part of the meal in a specific order and monitor them simultaneously, so that you can serve everything at the same time. Otherwise, your food will be cold, or even worse, burnt. So you choose the most efficient order to cook your meal.”

This activity should be done as a class, one step at a time. The essential mathematics is not complicated, but there is easily room for confusion. In particular, use the answers below to clarify any difficulties with Question #2 by using the answer for Question #3.

Student Page Answers:
1. *Answers may vary. The naïve answer is to just add up all the times, yielding 117 minutes. An excellent intuitive answer is to realize that of all the dishes, the mashed potatoes will take the most time. This in turn could permit a student to come up with the actual answer of 33 minutes. Should this happen, point out that preparing this meal is a much simpler project than building a space shuttle.* **2.** *See completed table and graph that accompanies the answer for #5.*
3. *E cannot start until both C(8) and D(5) are completed. It takes longer to peel and cut the yams than to melt the butter and brown sugar. The same is true for H, with paths from F(7) and G(6); K, with paths from I(8) and J(10); and again at N, with paths from B(17), E(28), H(13), L(30) and M(5).* **4.** *The critical path is: Start-J-K-L-N-Eat (33 minutes), which means that the mashed*

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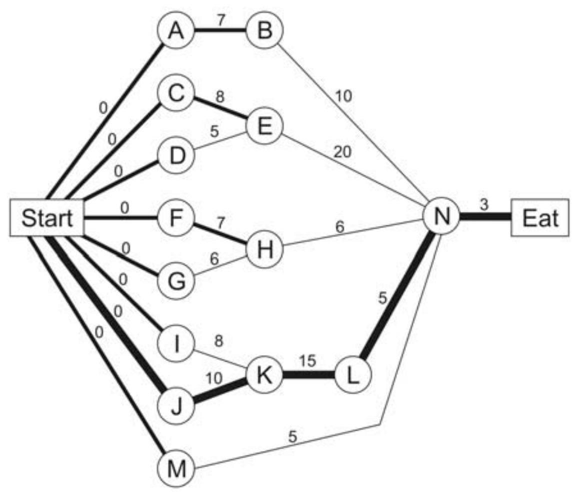
potatoes must be started first, since they take the longest time. **5.** *See the numbers with * on chart below. For these numbers, EST matches LST because these are on the critical path.*

Tasks

	Start	A	B	C	D	E	F	G	H	I	J	K	L	M	N	Eat
EST	0	0	7	0	0	8	0	0	7	0	0	10	25	0	30	33
LST	0*	13	20	2	5	10	17	18	24	2	0*	10*	25*	25	30*	33*

On the graph, all edges used to determine ESTs are in bold. The critical path is in “extra” bold.



6. *Since the critical path is preparing the mashed potatoes, and their total completion time is 33 minutes, they should be started at 5:27 (minute 0). The yams should start by minute 2, or 5:29. Next is the ham at minute 13, or 5:40. Next are the green beans at minute 17, or 5:44. Finally, setting the table can start as late as minute 25, or 5:52. This gets all preparations finished at minute 30, or 5:57, just in time to complete task N (plating and serving), to be ready to eat exactly at 6:00.* **7.** *Answers will vary. This is actually a very difficult problem to solve. In fact, there is no known efficient way to solve it in the general case, and this is addressed in the “Extensions.” Certainly it requires at least one cook and not more than five, since there are only five sub-tasks. One way to think about it is as follows. There are two types of activities, those that require constant attention by the cook, such as slicing the ham, and those that just require attention at the beginning of the task, such as heating a pot of water. The tasks requiring constant attention are A, C, D, F, I, L, M, and N, and their combined time equals 49 minutes. So, it is not possible for one person to prepare this meal in 33 minutes. In fact, it is not even possible for two cooks. It takes at least three. Tasks C, D, and I each require constant attention and all must be completed by minute 10. Together, they take 8 + 8 + 5 = 21 minutes, and no one person can do any two of these tasks by minute 10. With three cooks, it is fairly easy to schedule the remaining tasks. For example:*

Cook #1: Start J at 0, do I from 2 to 10, start K at 10, do G from 11 to 17, do L from 25 to 30, do N from 30 to 33.

Cook #2: Do C from 2 to 10, start E at 10, do A from 11 to 18,start B at 18, do M from 19 to 24.

Cook #3: Do D from 5 to 10, do F from 10 to 17, start H at 17.

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Name: ________________________________ Date: _______________

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In "End of Watch," Charlie is trying to understand the possible paths taken by a murder suspect. He initially considers using Ant Colony Optimization but he later realizes that he was only thinking of the paths of the suspect, and not the paths of the victim. He refers to and discusses Critical Path Analysis (CPA) as a mathematical way of finding the most efficient way to complete a set of tasks. In the case of the victim, it was the timeline for the day he was killed.

Although Charlie uses CPA to analyze the activities of the victim, he gives the example of preparing a holiday meal. According to Charlie, "You need to prepare and cook each part of the meal in a specific order and monitor them simultaneously, so that you can serve everything at the same time. Otherwise, your food will be cold, or even worse, burnt. So you choose the most efficient order to cook your meal."

Sometimes identifying all of the individual tasks can be the hardest part. Suppose that the following table summarizes all the tasks for preparing this meal:

Task label	**Task**	**Time (minutes)**	**Prerequisite**
Start	Decide to begin	0	None
A	Slice ham	7	None
B	Heat ham in microwave	10	A
C	Peel and cut yams	8	None
D	Melt butter and brown sugar	5	None
E	Cook yams	20	C, D
F	Rinse and trim green beans	7	None
G	Heat water for green beans	6	None
H	Cook green beans	6	F, G
I	Peel and cut potatoes	8	None
J	Heat water for potatoes	10	None
K	Cook potatoes	15	I, J
L	Mash and season potatoes	5	K
M	Set table	5	None
N	Plate and serve food	3	B, E, H, L, M
Eat	End of preparation	0	N

("Prerequisite" refers to the last task(s) that must finish before this one can start.)

1. Before continuing, determine what you think is the minimum completion time for preparing the meal. Explain your reasoning.

Using the table, the next step is to organize all of the tasks. One way to do this is to make a directed graph (the edges are one-way).

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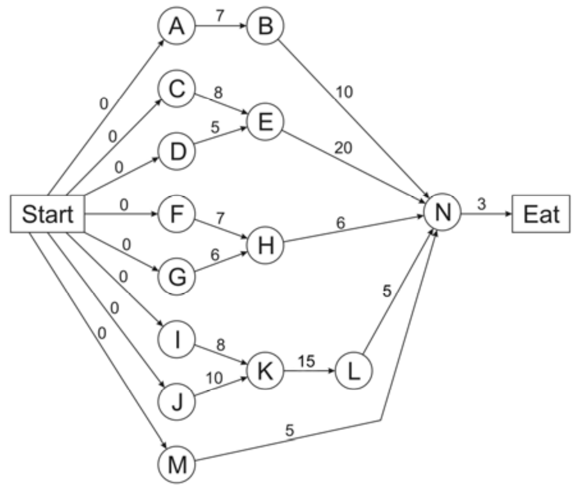
Each vertex represents a task. Each edge connects a task to the task(s) that directly follow it. The value of each edge that leads out of each vertex (called its weight) is the time it takes to complete each task.



2. Beginning at Start, trace through the graph to find the earliest starting time (EST) ("Start" is time = 0) for each task. That is, find the time required to complete all directly prerequisite tasks. Record each time in the table that follows Question #4. On the graph, highlight or otherwise mark each edge used for an EST. Note that task E can start at minute 8.

3. Why is 8 the EST for task E, and not 5? (The answer for this question is essential to determining the EST for tasks H, K and especially N.)

Identify the single path you highlighted that goes all the way from Start to Eat. This is the *critical path*, and its total time represents the minimum completion time for the meal.

4. What are the tasks on the critical path, and what is its total time? How does this compare with your answer for Question #1?

Tasks

	Start	A	B	C	D	E	F	G	H	I	J	K	L	M	N	Eat
EST						8										
LST																

5. Work backward through the graph from Eat to determine the latest starting time (LST) for each task *not* on the critical path. In other words, determine the latest time at which each task must start to ensure that all preparations finish at the same time.

6. This meal consists of five sub-projects (paths) from Start to Eat: the ham, the yams, the green beans, the mashed potatoes, and setting the table. If dinner is to be served at precisely 6:00 PM, what time should each sub-project start?

7. Do you think it is possible for this meal to be prepared in the minimum completion time entirely by just one person? If not, what is the minimum number of cooks needed? Either way, make up a schedule of tasks that completes the meal in the minimum time.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive mathematical topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Introduction

The last question in this activity leads to a different mathematical problem called the "machine-scheduling problem." In this activity the cooks are the "machines." Although there are algorithms to solve the problem, they include several assumptions, the first one being that when a machine starts a task, it does nothing else until the task is completed. These algorithms are also based on deciding on the number of machines *first*.

In the case of the holiday dinner, it is certainly reasonable that a cook could start a task (like putting water on to boil) then do something else and come back when the water is ready. The problem with this is that simply moving back and forth between tasks takes time, which will increase the total project time. There are also many human variables that must also be taken into account, like strength and fatigue, handling different food without washing hands first, etc. This means that there could be many possible solutions for the "holiday meal preparation problem."

For the Student

The other assumptions in the machine-scheduling problem are:

- If a task and machine are available, the machine will start on it immediately.
- The tasks have been arranged by prerequisites, as in this activity.
- The tasks are also listed in their order of their importance, independent of the order in which they must be done.

Among the algorithms to solve machine-scheduling problems are:

- List processing: at any time, assign to the lowest numbered machine the first task on the importance list that has all of its prerequisites finished.
- Decreasing time: use the list-processing algorithm, but arrange the tasks in decreasing order to time required to complete them.
- Longest-path: Any time a machine is free, it starts the available task that has the longest path from the end of the project.

Using the above assumptions, try these algorithms on the holiday meal problem for one, two, or more cooks.

Additional Resources

For lessons on CPA and machine-scheduling problems appropriate for many high school students, see Chapter 3 of For All Practical Purposes, W.H. Freeman, NY, 2006.

The American Mathematical Society provides a comprehensive overview of these problems, along with the more general problem called "bin packing" (like scheduling TV commercials during program breaks), at:
http://www.ams.org/featurecolumn/archive/packings1.html

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