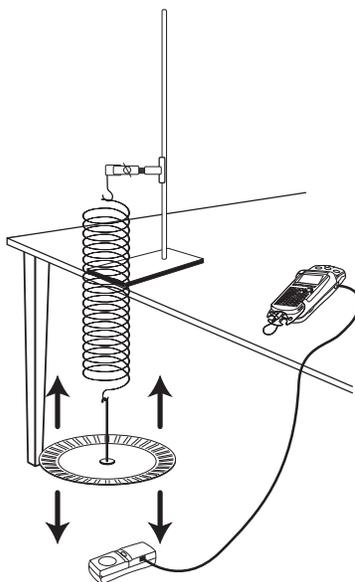


# Up and Down: Damped Harmonic Motion

An object hanging from a spring can bounce up and down in a simple way. The vertical position of the object can be described mathematically in terms of a simple sinusoidal equation. In the real world, however, resistive forces such as friction are always present and cause the object to slow down. This effect is called *damping*.

Most oscillating objects experience damping and move in a modified periodic manner so that the amplitude gets smaller and smaller with each cycle. Common examples of damped oscillators include an empty rocking chair as it comes to rest after being pushed and a vibrating diving board after a swimmer leaves it. At first, the problem of modeling this type of motion with a mathematical equation may seem extraordinarily complex. Surprisingly, it can be analyzed rather thoroughly using basic math concepts with which you are already familiar.

In this activity, you will collect motion data as a paper plate attached to a light spring oscillates up and down above a Motion Detector. Then, you will find an appropriate mathematical model for the resulting data set.



## OBJECTIVES

- Record the motion data for a plate bouncing at the end of a light spring.
- Analyze the motion data to determine frequency, period and amplitude information.
- Model the oscillatory part of the data using trigonometric functions.
- Model the damping using an exponential function.
- Create a composite model of damping and oscillation.
- Compare the composite model to experimental data.

## MATERIALS

TI-83 Plus or TI-84 Plus graphing calculator  
 EasyData application  
 CBR 2 or Go! Motion and direct calculator cable  
 or Motion Detector and data-collection interface

ring stand  
 standard slotted mass hanger  
 Slinky™ or light spring  
 paper plate

## PROCEDURE

1. Set up the Motion Detector and calculator.
  - a. Open the pivoting head of the Motion Detector. If your Motion Detector has a sensitivity switch, set it to Normal as shown.
  - b. Turn on the calculator and connect it to the Motion Detector. (This may require the use of a data-collection interface.)
2. Position the Motion Detector on the floor so that the head is pointing straight upward.
3. Suspend the spring and its plate over the Motion Detector from a ring stand or other fixed support. Be sure that no table edge or other barrier is near the path between the plate and the Motion Detector. Wait until the plate and spring stop moving. The distance to the Motion Detector should be at least 70 cm. This position is called the *equilibrium* position.
4. Set up EasyData for data collection.
  - a. Start the EasyData application, if it is not already running.
  - b. Select **[File]** from the Main screen, and then select **New** to reset the application.
  - c. Select **[Setup]** from the Main screen, then select **Time Graph...**
  - d. Select **[Edit]** on the Time Graph Settings screen.
  - e. Enter **0.1** as the time between samples in seconds.
  - f. Select **[Next]**.
  - g. Enter **100** as the number of samples and select **[Next]**.
  - h. Select **[OK]** to return to the Main screen.
5. So that the zero reference position will be the equilibrium position of the plate, zero the Motion Detector.
  - a. Select **[Setup]** from the Main screen, and then select **Zero...**
  - b. Confirm that the plate is still stationary. Select **[Zero]** to zero the Motion Detector.
6. Pull the plate downward about 10 cm, and allow it to oscillate up and down. Select **[Start]** to begin data collection. Data collection will run for ten seconds.
7. When data collection is complete, a graph of distance versus time will be displayed. Examine the distance versus time graph. The graph should contain a sine curve of diminishing amplitude. Check with your instructor if you are not sure whether you need to repeat the data collection. To repeat data collection, select **[Main]** and repeat Step 6.
8. Once you are satisfied with your data, select **[Main]** to return to the Main screen. Exit EasyData by selecting **[Quit]**, and then selecting **[OK]**.



## ANALYSIS

1. Redisplay the graph outside EasyData.
  - a. Press  $\text{2nd}$  [STAT PLOT].
  - b. Press  $\text{ENTER}$  to select Plot1 and press  $\text{ENTER}$  again to select On.
  - c. Press  $\text{ZOOM}$ .
  - d. Press  $\text{V}$  until ZoomStat is highlighted; press  $\text{ENTER}$  to display a graph with the  $x$  and  $y$  ranges set to fill the screen with data.
  - e. Press  $\text{TRACE}$  to determine the coordinates of a point on the graph using the cursor keys.
2. The graph shows several features, each of which you will model with an appropriate mathematical function.

First, notice the envelope of the back-and-forth, or *oscillatory*, motion. The amplitude of the motion decreases in time in a regular way. You will model the time variation of the amplitude with one function. This oscillatory part of the motion can be modeled with a cosine function  $y = A\cos(B(x - C))$ , where

- $x$  is time in seconds,
- $y$  is distance in meters, and
- $A$  is the amplitude of the motion, or the largest distance above the center position.

To estimate the amplitude parameter  $A$ , trace across the graph using the  $\text{D}$  key to the first positive maximum. Record the  $y$  value as  $A$  in the Data Table on the *Data Collection and Analysis* sheet. Record this, and all distance values, to three significant figures.

3. The shortest time it takes for a repetitive motion to repeat itself is called the *period*. One way to find the period of the plate is to find the time between two successive maximum or minimum points. Continue to trace across the graph to determine the  $x$  values for two adjacent maximum or minimum peaks. Record these two values as  $t_1$  and  $t_2$ . Calculate the difference, which is the period  $T$ , and record it in the Data Table.
4. The parameter  $C$  represents the horizontal shift of the cosine function. Since the cosine function has a maximum value when its argument is zero, the time when the data first reaches a maximum will serve as an estimate for  $C$ . Trace to the first maximum and note the time ( $x$ ) value. Record this value as  $C$  in the Data Table.
5. To compare the model to the data, enter the model equation into the calculator.
  - a. Press  $\text{Y=}$ .
  - b. Press  $\text{CLEAR}$  to remove any existing equation.
  - c. Enter the expression  $A*\cos(B*(X-C))$  in the  $Y_1$  field.
  - d. To make the model equation plot with a bold line, press  $\text{F5}$  until the diagonal line to the left of  $Y_1$  is highlighted. Press  $\text{ENTER}$  until the line is shown as a bold line.
  - e. Press  $\text{2nd}$  [QUIT] to return to the home screen.
6. The parameter  $A$  represents the amplitude of the waveform. Enter your value for the parameter  $A$  in your calculator's memory. To do this, enter the value for the amplitude from the Data Table. Press  $\text{STO}$  A  $\text{ENTER}$  to store the value in the variable  $A$ .
7. The sinusoidal model has a parameter  $B$  that represents the number of cycles the sinusoidal function makes during the natural period of the cosine function.

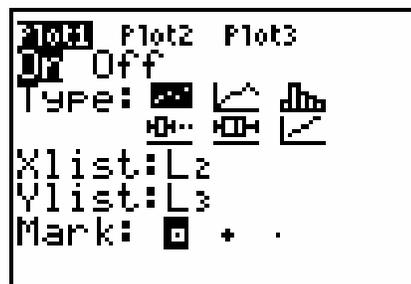
## Activity 27

Find  $B$  by taking  $2\pi$  (the natural period of the cosine function) divided by the period  $T$  of the waveform (the time for one cycle). Store this value in the calculator after performing the division by pressing  $\text{STO} \rightarrow B \text{ ENTER}$  to store the value in the variable  $B$ . Also record the value in the Data Table.

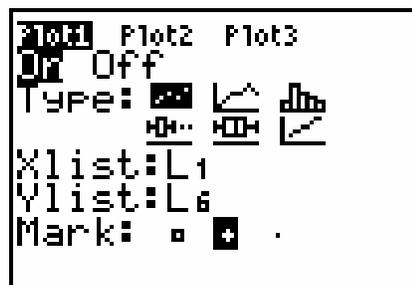
8. Enter the value of  $C$  from the Data Table, and press  $\text{STO} \rightarrow C \text{ ENTER}$  to store the value in variable  $C$ .
9. You have now entered the model equation and values for the three parameters  $A$ ,  $B$ , and  $C$ . Press  $\text{GRAPH}$  to display a graph of the motion data with the cosine model equation.

⇒ Answer Question 1 on the *Data Collection and Analysis* sheet.

10. Now that you have modeled the oscillatory part of the motion, turn off the plotting of the cosine model.
  - a. Press  $\text{Y=}$ .
  - b. Press  $\leftarrow$  so that the  $=$  sign is highlighted; press  $\text{ENTER}$  to turn off plotting of the equation.
  - c. Press  $\text{2nd}$  [QUIT] to return to the home screen.
11. To model the time variation of the amplitude, you need to record the maximum values and their times in a temporary table of your own design.
  - a. Press  $\text{GRAPH}$  and then press  $\text{TRACE}$  to determine the coordinates of a point on the graph using the cursor keys.
  - b. Record the  $x$  and  $y$  values of each maximum peak to three significant figures. Continue to trace and record until you have reached the end of the graph or until you reach six points.
12. To plot the points you just recorded, you need to enter them into new lists in the calculator.
  - a. Press  $\text{STAT}$  and press  $\text{ENTER}$  to see the data lists.
  - b. Lists 2 and 3 are available for you to use. Use the cursor keys to highlight the  $L_2$  header, and press  $\text{CLEAR}$  to clear the list.
  - c. Press  $\text{ENTER}$  to move to the first element of  $L_2$  and then enter all of the time values, ending each entry with  $\text{ENTER}$ .
  - d. Use  $\rightarrow$  and  $\uparrow$  to move to the header of  $L_3$ . Press  $\text{CLEAR}$  to clear the list.
  - e. Press  $\text{ENTER}$  to move to the first element of  $L_3$ .
  - f. Enter the maximum distance data values, ending each entry with  $\text{ENTER}$ . You must have the same number of elements in  $L_2$  as in  $L_3$  when you are done. If not, check the entries for errors and correct them.
13. Now that you have the data to be plotted stored in lists in the calculator, you are ready to plot just the peak values of the distance and time data.
  - a. Press  $\text{2nd}$  [STAT PLOT] and press  $\text{ENTER}$  to select Plot 1.
  - b. Change the Plot1 settings to match the screen shown here. Press  $\text{ENTER}$  to select any of the settings you change.
  - c. Press  $\text{ZOOM}$  and then select ZoomStat (use cursor keys to scroll to ZoomStat) to draw a graph with the  $x$  and  $y$  ranges set to fill the screen with data.



14. The graph should show a gradual decay in the amplitude of the oscillations of the plate. To incorporate the decay into a model for the data, fit an exponential function to the amplitude curve.
- Press  $\text{STAT}$  and use the cursor keys to highlight CALC.
  - Press the number adjacent to ExpReg to copy the command to the home screen.
  - Press  $\text{2nd}$  [L2]  $\text{,}$   $\text{2nd}$  [L3]  $\text{,}$  to enter the lists containing the data.
  - Press  $\text{VAR}$  and use the cursor keys to highlight Y-VARS.
  - Select Function by pressing  $\text{ENTER}$ .
  - Press  $\text{V}$  and  $\text{ENTER}$  to copy Y2 to the expression.
  - On the home screen, you will see the entry ExpReg L2, L3, Y2. This command will perform an exponential regression with L2 as the  $x$  and L3 as the  $y$  values. The resulting regression equation will be stored in equation variable Y2. Press  $\text{ENTER}$  to perform the regression.
  - Press  $\text{GRAPH}$  to see the graph.
15. The new graph shows the maximum amplitude points and the fitted exponential function. To combine the exponential decay (often known as the *envelope* of the oscillations) with the oscillatory cosine model, you can define a new function that is the product of the cosine model and the exponential function. The cosine model includes the parameter  $A$  to represent the amplitude; the Y2 exponential function can replace the amplitude parameter to improve the model function.
- Press  $\text{Y=}$ .
  - The cursor will be flashing on the  $A$  variable in Y1. Press  $\text{VAR}$  and press  $\text{D}$  to highlight the Y-VARS menu.
  - Press  $\text{ENTER}$  to display the function menu.
  - Select Y2 to paste the Y2 function to the equation screen, replacing the  $A$  variable.
  - Press  $\text{V}$  until you highlight the Y2 function. Since you no longer need this part plotted, press  $\text{L}$  until the = sign is flashing, and press  $\text{ENTER}$  to disable plotting of Y2.
  - Record the composite model equation in the Data Table on the *Data Collection and Analysis* sheet, including both the exponential and oscillating parts.
16. Now you can view a graph of the final model plotted with the complete experimental data.
- Press  $\text{2nd}$  [STAT PLOT] and press  $\text{ENTER}$  to select Plot 1.
  - Change the Plot1 settings to match the screen shown here. Press  $\text{ENTER}$  to select any of the settings you change.
  - Press  $\text{ZOOM}$  and then select ZoomStat (use cursor keys to scroll to ZoomStat) to draw a graph with the  $x$  and  $y$  ranges set to fill the screen with data.



⇒ Answer Questions 2–4 on the *Data Collection and Analysis* sheet.

## EXTENSIONS

1. What would a plot of velocity versus time look like for the damped oscillations you studied in this activity? Create a scatter plot of velocity versus time.
2. Use the numerical computation tools of the calculator to find a mathematical equation that models velocity as a function of time. How well does the modeling curve fit the data?
3. What would a plot of velocity versus position look like for a damped oscillator? Make a sketch of your prediction, and then create a scatter plot of velocity versus position. How can the mathematical models you created for position and velocity as functions of time be used to model the velocity-position data plot?

**Hint:** You will need to set the calculator to *parametric* mode.



