**Concepts**

If a function  has derivatives of all orders, then under certain conditions we can write  as a Taylor series centered at :



In the case where , the Taylor series becomes



This is called a Maclaurin series.

This expression means that  is the limit of the sequence of partial sums. The partial sums are



The expression for  is a polynomial of degree .  is called the th-degree Taylor polynomial of  at , or centered at .

Let  so that , then  is called the remainder of the Taylor series. If it is possible to show that  then



**Course and Exam Description Unit**

Section 10.11: Finding Taylor Polynomial Approximations of Functions

Section 10.14: Finding Taylor or Maclaurin Series for a Function

**Calculator Files**

Taylor\_Polynomial\_Examples.tns

**Using the Document**

This tns file is used to produce the graphs of various Taylor polynomials, . These graphs are used to study how the accuracy of a Taylor polynomial is associated with the degree of the Taylor polynomial. The accuracy of each Taylor polynomial is visualized and can be related to symmetry, arc length, and any points of discontinuity.

Five common functions are considered and some specific questions about associated Taylor polynomials are included.

Page 1.1

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|  | This opening page lists the five functions considered in this tns file: , , , , and . Note that a CAS calculator is not required for any of these examples. |

Page 2.1

|  |  |
| --- | --- |
|  | This page describes the first example, the notation used, and the method for changing the values of  and . In Problem 2 of this tns file,  and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value. |

Page 2.2

|  |  |
| --- | --- |
|  | The graph of  is black and the graph of  is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are  and .  Note that it is also possible to grab the point  on the  -axis and move it horizontally to change the value. |

**Problems**

1. For , sketch and describe the graph of .
2. Use the graph of  and the Trace All feature to describe the accuracy of the Taylor polynomial approximation as  moves farther away from .
3. Set . Sketch and describe the graph of .
4. Set . Sketch and describe the graph of .
5. Consider the graph of other Taylor polynomials for . Describe the accuracy of the Taylor polynomial approximation as  increases.

Page 2.3

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| --- | --- |
|  | This is a Lists and Spreadsheet page in which values for  can be entered in column A. The following values are automatically computed: , , and  (an indication of the accuracy of the approximation), in columns B, C, and D, respectively. These values are dependent on the current values of  and . |

**Problems**

Change the values of  and  as necessary and use Page 2.3, the Lists and Spreadsheet page, to answer the following questions.

1. For a fixed value of , describe the accuracy of the Taylor polynomial approximation as the values of  are farther away from .
2. For fixed values of  and , describe the accuracy of the Taylor polynomial approximation as  increases.
3. Set . For , find an interval in which the Taylor polynomial is a good approximation for .

Note: Page 2.4 is used for background calculations. The equations and entries on this Lists and Spreadsheet page should be left unchanged to ensure the accuracy of the results presented on other pages of this problem.

Page 3.1

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|  | This page describes the second example, the notation used, and the method for changing the values of  and . In Problem 3 of this tns file,  and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value. Note that the value of  must be greater than 0 in this example. |

Page 3.2

|  |  |
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|  | The graph of  is black and the graph of  is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are  and .  Note that it is also possible to grab the point  on the  -axis and move it horizontally to change the value. |

**Problems**

Change the values of  and  as necessary and use Page 3.2 to answer the following questions.

1. For , describe the accuracy of the Taylor polynomial approximation as  increases.
2. Describe the behavior of each Taylor polynomial as  and as . Explain how the graph of the Taylor polynomial changes, as , as  increases by 1, for example, from  to . Explain why this property of the Taylor polynomials alternates as  increases.
3. Set . Consider the graph of  for various values of . Explain why the Taylor polynomial appears to be a very good approximation to the left of  but diverges rapidly to the right of .

Page 4.1

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|  | This page describes the third example, the notation used, and the method for changing the values of  and . In Problem 4 of this tns file,  and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value.  Note that in this example, the value of  must be in the interval and can be changed in increments of . |

Page 4.2

|  |  |
| --- | --- |
|  | The graph of  is black and the graph of  is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are  and .  Note that it is also possible to grab the point  on the  -axis and move it horizontally to change the value. |

Page 4.3

|  |  |
| --- | --- |
|  | This Notes page defines and displays the Taylor polynomial  for the current values of  and . |

**Problems**

Change the values of  and  as necessary and use Pages 4.2 and 4.3 to answer the following questions.

1. Set  and . Describe the graph of the Taylor polynomial . Find the Taylor polynomial and describe the approximation for  for  close to .
2. Set . Consider the graph of the Taylor polynomial  as  increases. Explain why the graph of the Taylor polynomials for  and  are identical, and for  and , etc.
3. For each value of  and , describe the accuracy of the Taylor approximation about the point

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Page 5.1

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|  | This page describes the fourth example, the notation used, and the method for changing the values of  and . In Problem 5 of this tns file,  and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value.  Note that in this example, the value of  must be in the interval and can be changed in increments of . |

Page 5.2

|  |  |
| --- | --- |
|  | The graph of  is black and the graph of  is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are  and .  Note that it is also possible to grab the point  on the  -axis and move it horizontally to change the value. |

Page 5.3

|  |  |
| --- | --- |
|  | This Notes page defines and displays the Taylor polynomial  for the current values of  and . |

**Problems**

Change the values of  and  as necessary and use Pages 5.2 and 5.3 to answer the following questions.

1. Set  and . Describe the graph of the Taylor polynomial. Find the Taylor polynomial and explain why the slope of this linear approximation is 0.
2. Set . Consider the graph of the Taylor polynomial  as  increases. Explain why the graph of the Taylor polynomials for  and  are identical, and for  and for , etc.

Page 6.1

|  |  |
| --- | --- |
|  | This page describes the fifth example, the notation used, and the method for changing the values of  and . In Problem 6 of this tns file,  and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value.  Note that in this example, . |

Page 6.2

|  |  |
| --- | --- |
|  | The graph of  is black and the graph of  is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are  and .  Note that it is also possible to grab the point  on the  -axis and move it horizontally to change the value. |

Page 6.3

|  |  |
| --- | --- |
|  | This Notes page defines and displays the Taylor polynomial  for the current values of  and . |

**Problems**

Change the values of  and  as necessary and use Pages 6.2 and 6.3 to answer the following questions.

1. Set . Consider the graph of the Taylor polynomial  for various values of . Explain why there is no graph of the Taylor polynomial to the right of .
2. Set  and . Explain the accuracy of the Taylor polynomial. Explain the accuracy of the Taylor polynomial . Why does the Taylor polynomial appear to be a much better approximation to the right of  than to the left?
3. Explain how to obtain the graph of a Taylor polynomial that can be used to approximate the portion of the graph of  to the right of .

**Suggested Extensions**

1. Explain how the accuracy of a Taylor polynomial is related to the degree of the Taylor polynomial and the value of .
2. Describe the interval about  on which a Taylor polynomial is fairly accurate.
3. Suppose the function  has a discontinuity at . Explain how this value affects a Taylor polynomial for .
4. What is the relationship between the th derivative of the function  and the th derivative of the corresponding Taylor polynomial ?
5. Consider exploring the Taylor polynomials associated with the following functions.
6. 
7. 
8. 
9. 
10. 